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QUARTERLY OF APPLIED MATHEMATICS

Vol. XI JANUARY, 1954 No. 4

# SIMILARITY LAWS FOR SUPERSONIC FLOWS*

BY

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**Summary.** The non-linear differential equation for the velocity potential of three-dimensional steady irrotational supersonic flow past wings of finite span has been investigated. It is found that the whole Mach number range from 1 to $\infty$ may be divided into two regions (not strictly divided), in each of which similarity laws are obtained, with two parameters $K_1 = (M^2 - 1)^{1/2}/\tau^n$ and $K_2 = A(M^2 - 1)^{1/2}$; $\tau$ is the non-dimensional thickness ratio, $A$ the aspect ratio of the wing, $M$ the Mach number of the uniform stream in which the wing is placed. The factor $n$ is given explicitly as a function of $M$ and $\tau$; in the lower region of Mach numbers it tends to 1/3 as $M \to 1$, for all $\tau$, giving the ordinary transonic rule, and in the upper region it tends to $-1$ as $M \to \infty$, for all $\tau$, as in the ordinary hypersonic rule.

It is shown that both two-dimensional flow and flow over a three-dimensional slender body, including axially symmetrical flow, are special cases of the present analysis, involving only one parameter $K_1$ in the similarity rules.

**I. Introduction.** One of the major difficulties in solving problems of flow of compressible fluid is the non-linearity of the differential equations that govern the flow. Over a certain range of Mach number, the differential equations of flow of compressible fluid can be linearized for many investigations of practical importance, and the resulting equations give valuable information on the flow field of a compressible fluid. However, there exist other ranges of Mach numbers, in particular the transonic flow ($M \cong 1$) and the hypersonic flow ($M \gg 1$), where the differential equations cannot be linearized. In these cases, we have to study non-linear equations. At the present time, there is no general method of solving these non-linear differential equations. For practical purposes the study of the effects of non-linearity on the flow must rely mainly on experimental results. In order to increase the value of either theoretical or experimental data, it is useful to find some similarity laws which will enable a family of possible solutions to be deduced from a single result.

In general similarity laws may be found for bodies moving in the compressible fluid, which have at least one dimension perpendicular to the direction of main flow small in comparison with that in the direction of main flow. For the linearized theory well-known similarity laws have been obtained by Glauert [1] and Prandtl [2] for subsonic flow, and by Ackeret [3] for supersonic flow. However, it will be shown later that the similarity

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laws for linearized theory are arbitrary. This fact has already been observed in the study of linearized axially symmetrical flow of compressible fluid [4, 5], where contradictory results have been obtained by the use of different similarity laws. Unique similarity laws can be determined only when non-linear terms are considered.

Von Kármán [6] was the first to obtain the similarity laws for transonic flows, where the fluid velocity is very near to the velocity of sound, for both the two-dimensional and the axially symmetrical cases. Spreiter [7] extended the transonic similarity laws to wings of finite span.

The similarity laws for hypersonic flow, where the fluid velocity is much larger than the local velocity of sound, were first obtained by Tsien [8], again both for two dimensions and axial symmetry. Hayes [9] extended the hypersonic similarity laws to three-dimensional slender bodies.

Spreiter [7] tried to combine the similarity laws of transonic flow with those of linearized theory. He was able to do so because of the arbitrariness of the similarity laws of linearized theory. Van Dyke [10] empirically generalized the hypersonic similarity laws to the supersonic flow where the fluid velocity is not much larger than the local velocity of sound. It is the object of this paper to investigate the possibility of determining uniquely similarity laws which link flows over more widely extended ranges of Mach number than are covered by the particular cases cited above.

We begin with the non-linear differential equation for the velocity potential of three dimensional steady irrotational supersonic flow for wings of finite span; two-dimensional and axially symmetrical flows, and flow past three-dimensional slender bodies, may be considered as special cases of this general flow. The relative importance of the various non-linear terms is discussed so that the ordinary transonic and hypersonic similarity laws are generalized for large Mach number range. It is found that from a known solution for a given Mach number and ratio of thickness to chord, a family of solutions may be determined; there is however a barrier dividing the Mach number range into two regions, so that a solution which falls into the 'generalized transonic range' cannot be used to give a flow in the 'generalized hypersonic range', and vice-versa.

**2. Fundamental equations and boundary conditions.** If $\Phi$ is the velocity potential of a three-dimensional steady irrotational flow of compressible fluid, the differential equation for $\Phi$ is

$$(a^2 - \Phi_x^2)\Phi_{xx} + (a^2 - \Phi_y^2)\Phi_{yy} + (a^2 - \Phi_z^2(\Phi_{zz} - 2\Phi_x\Phi_y\Phi_{xy}$$

$$- 2\Phi_y\Phi_z\Phi_{yz} - 2\Phi_z\Phi_x\Phi_{zx} = 0, \qquad (1)$$

where $x$, $y$, $z$ are the Cartesian coordinates, subscripts denote partial derivatives, i.e., $\Phi_x = \partial\Phi/\partial x$ etc., and $a$ is the local sound speed which is determined by the equation:

$$a^2 + \frac{\gamma - 1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) = a_0^2 . \qquad (2)$$

Here $\gamma$ is the ratio of the specific heats and $a_0$ is the speed of sound in the gas at rest.

Now if a thin wing of finite span is placed in an otherwise uniform stream of velocity $V$ in the $x$-direction, we may introduce a perturbed velocity potential $\phi$ such that

$$\Phi = V(x + \phi) \qquad (3)$$

with

$$\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \ll 1.$$

Equation (2) may be written as follows:

$$\frac{a^2}{a_1^2} = 1 - \frac{\gamma - 1}{2} M^2(2\phi_x + \phi_x^2 + \phi_y^2 + \phi_z^2), \tag{4}$$

where $a_1$ is the speed of sound corresponding to the free stream velocity $V$ and $M = V/a_1$, the Mach number in the undisturbed stream.

Substituting equations (3) and (4) into (1) and retaining terms up to second order, one has, with $\lambda^2 = (\gamma - 1)/(\gamma + 1)$,

$$\begin{aligned}\left\{1 - M^2 - (\gamma + 1)M^2\phi_x - \frac{\gamma - 1}{2} M^2(\phi_y^2 + \phi_z^2)\right\}\phi_{xx} \\ + \left\{1 - (\gamma - 1)M^2\phi_x - \frac{\gamma + 1}{2} M^2(\phi_y^2 + \lambda^2\phi_z^2)\right\}\phi_{yy} \\ + \left\{1 - (\gamma - 1)M^2\phi_x - \frac{\gamma + 1}{2} M^2(\lambda^2\phi_y^2 + \phi_z^2)\right\}\phi_{zz} \\ - 2M^2\phi_y\phi_{xy} - 2M^2\phi_z\phi_{xz} = 0.\end{aligned} \tag{5}$$

This is the fundamental differential equation for $\phi$ for the consideration of similarity laws, and is a generalization of Tsien's equation for two-dimensional flow [8].

In (5) we have retained terms which, while not important everywhere, may be so in certain regions. For example, in hypersonic flow $\phi_y^2$, $\phi_z^2$, $\phi_x$ are all of the same order of magnitude. Furthermore, while we retain in (5) terms which are usually second-order, we shall seek and use those terms which give an effectively first-order equation.

The first-order boundary conditions for $\phi$ are:

(1) at infinity

$$\phi_x = \phi_y = \phi_z = 0; \tag{6}$$

(2) on the surface of the wing, which is represented by

$$z = h(x, y), \tag{7}$$

the normal velocity component is zero, i.e.

$$\left(\frac{\partial \phi}{\partial z}\right)_{z=0} = \frac{\partial h(x, y)}{\partial x} \tag{8}$$

**3. General discussion of similarity rules.** We are going to find conditions under which it is possible to reduce the differential equation for $\phi$ and the boundary conditions simultaneously into non-dimensional form. We write

$$x = \xi c, \qquad y = \eta b, \qquad z = \zeta l, \qquad \phi = \phi' m, \tag{9}$$

where $c$ is the mean chord of the wing and $b$ its span; thus, $A = b/c =$ aspect ratio. $l$ and $m$ are conversion factors which are to be determined.

Substituting equation (9) into (6) and (8), we have,

$$\phi'_\xi = \phi'_\eta = \phi'_\zeta = 0 \qquad \text{at infinity,} \tag{10}$$

$$\left(\frac{\partial \phi'}{\partial \zeta}\right)_{\zeta=0} = \frac{l\tau}{m} f_\xi(\xi, \eta), \tag{11}$$

where $Tf(\xi, \eta) = h(x, y)$ and $T = \tau c$ is the maximum thickness of the airfoil section; it will be supposed that $\tau \ll 1$.

Substituting (9) into (5), we have

$$\begin{aligned}
&\left[(M^2 - 1)\left(\frac{l}{c}\right)^2 + (\gamma + 1)M^2\left(\frac{m}{c}\right)\left(\frac{l}{c}\right)^2 \phi'_\xi + \frac{\gamma - 1}{2} M^2\left(\frac{m}{c}\right)^2\left(\frac{l}{c}\right)^2\left(\frac{\phi'^2_\eta}{A^2} + \phi'^2_\zeta\left(\frac{c}{l}\right)^2\right)\right]\phi'_{\xi\xi} \\
&\quad + \left[-1 + (\gamma - 1)M^2 \frac{m}{c}\phi'_\xi + \frac{\gamma + 1}{2} M^2\left(\frac{m}{c}\right)^2\left\{\frac{\phi'^2_\eta}{A^2} + \lambda^2\left(\frac{c}{l}\right)^2\phi'^2_\zeta\right\}\right]\phi_{\eta\eta}\left(\frac{l}{cA}\right)^2 \\
&\quad + \left[-1 + (\gamma - 1)M^2 \frac{m}{c}\phi'_\xi + \frac{\gamma + 1}{2} M^2\left(\frac{m}{c}\right)^2\left\{\lambda^2\frac{\phi'^2_\eta}{A^2} + \left(\frac{c}{l}\right)^2\phi'^2_\zeta\right\}\right]\phi'_{\zeta\zeta} \\
&\quad + 2M^2\left(\frac{m}{c}\right)\left(\frac{l}{cA}\right)^2\phi'_\eta\phi'_{\xi\eta} + 2M^2\left(\frac{m}{c}\right)\phi'_\zeta\phi'_{\zeta\xi} = 0.
\end{aligned} \tag{12}$$

In order to get the parameters in the similarity laws we write

$$l = c\tau^{-n}, \qquad m = c\tau^{n'}, \tag{13}$$

where $n$ and $n'$ are factors to be determined.

Then, for similarity laws to be possible, considering only the linear terms in (12), the following necessary conditions are found:

$$\frac{M^2 - 1}{\tau^{2n}} = K_1^2\,, \tag{14}$$

$$A\tau^n = K'_2\,, \tag{15}$$

or, combining these,

$$A(M^2 - 1)^{1/2} = K_2 \tag{16}$$

where $K_1$, $K'_2$ and $K_2$ are constants.

From the boundary conditions, we have

$$\tau^{1-n-n'} = \text{constant}; \tag{17}$$

this can only be true for variable $\tau$ provided that

$$n + n' = 1, \tag{18}$$

and the value of the constant is unity.

Equations (14), (16) and (18) represent the conditions for similarity laws in the linearized theory of compressible flow. Only the parameter $K_2$ for the aspect ratio (16) is unique; the index $n$ in the parameter $K_1$, is arbitrary, since the choice of $n'$ is arbitrary. In order to have unique similarity laws it is necessary to study the non-linear terms. From (12) we see that the non-linear terms fall into two groups, one group being the more important for transonic-supersonic flow ($n > 0$), the other for supersonic-hypersonic flow ($n < 0$). In the transonic-supersonic flow region the important non-linear term is $(\gamma + 1)M^2(m/c)(l/c)^2\phi'_\xi$. For this term to be unchanged with variation of airfoil

thickness, for example, in the immediate transonic region ($M \cong 1$) we must have $\tau^{n'-2n} =$ const., i.e.,

$$n' - 2n = 0. \tag{19}$$

From (18) and (19), we have

$$n = \tfrac{1}{3}; \qquad n' = \tfrac{2}{3}. \tag{20}$$

This is the well-known transonic similarity law due to von Kármán [6].

When $M$ is very large, the important non-linear terms are in the second square bracket in (12); both are of the same order of magnitude as $(\gamma - 1)\ M^2(m/c)\ \phi'_\xi$; for similarity we require

$$M^2\tau^{n'} = \text{const}. \tag{21}$$

Thus, from (14) and (21),

$$n' = -2n. \tag{22}$$

Finally, from equations (18) and (22),

$$n = -1, \qquad n' = 2 \tag{23}$$

This is the well-known hypersonic similarity law due to Tsien.

It is clear, from an examination of (12), that in order to obtain similarity laws which are valid in the intermediate (supersonic) range of Mach numbers we must look for a variation of $n$ with Mach number. It is found in the next section that $n$ depends upon two parameters, $M$ and $\tau$ (say), but that the value of $n$ tends to $1/3$ as $M \to 1$ and to $-1$ as $M \to \infty$, for all $\tau$.

**4. Generalized similarity law for transonic-supersonic flow.** As $M$ increases from unity, the non-linear term of lowest order is the second term in the first square bracket in (12).

In order to preserve the form of (12) unchanged up to and including this term, when the Mach number is not restricted to be in the immediate neighborhood of unity, the parameters required are

$$\frac{(M^2 - 1)}{\tau^{2n}} = K_1^2\,, \qquad A(M^2 - 1)^{1/2} = K_2\,,$$

with $n + n' = 1$ from the boundary condition as before, and

$$M^2\tau^{n'-2n} = K_3^2\,. \tag{24}$$

By elimination of $\tau$, we derive

$$n[4 \log (K_3/M) + 3 \log \{(M^2 - 1)/K_1^2\}] = \log \{(M^2 - 1)/K_1^2\}. \tag{25}$$

This shows clearly that $n \to 1/3$ as $M \to 1$. Furthermore, by differentiating and proceeding to the limit, it may be shown that

$$\lim_{M\to 1(+0)} \left[\frac{dn}{dM}\right] = \begin{cases} 0 & (K_3 = 1), \\ \infty & (K_3 > 1), \\ -\infty & (K_3 < 1). \end{cases}$$

It is plausible to reject infinite values of $dn/dM$ at $M = 1$, in view of the established position of the transonic similarity law which requires $n = 1/3$ to be approximately valid in the neighborhood of $M = 1$. Then $K_3 = 1$, and the law of variation of $n$ with $M$ is

$$n = \frac{1}{3} \log \left(\frac{M^2 - 1}{K_1^2}\right) \Big/ \log \left(\frac{M^2 - 1}{K_1^2 M^{4/3}}\right). \tag{26}$$

Examples of the variation of $n$ with $M$, for three different values of the parameter $K_1$, are shown in Figure 1.

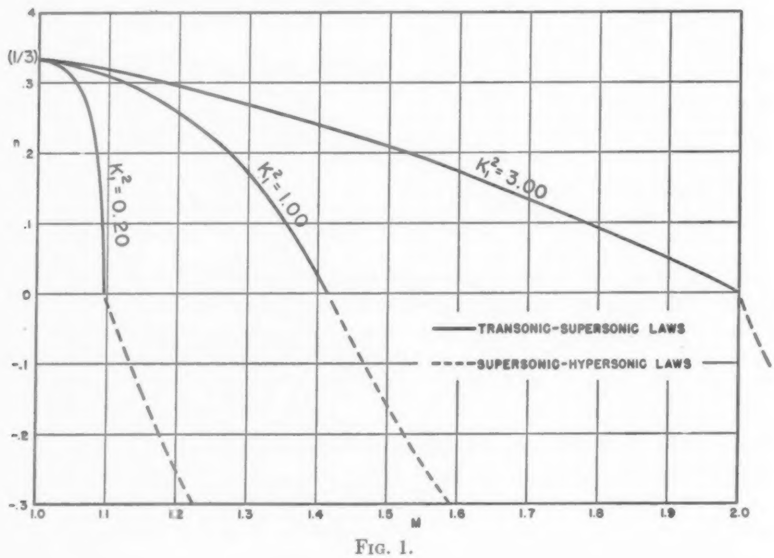


FIG. 1.

We are now able to deduce similarity laws in the sense that a solution $\phi(\xi, \eta, \zeta)$ of the differential equation of motion may be used to give an infinity of flows in the transonic-supersonic region. Given values of the parameters $K_1$, $K_2$, if we choose a value for $M$ the corresponding values of $\tau$, $A$, $n$ are found.

It will be noticed that the solid curves have been stopped at $n = 0$. As $n \to 0$ the relative importance of the non-linear terms in the second square bracket in (12) increases, and at $n = 0$ these terms are of the same order of magnitude as the "transonic" term which has been the crux of the above discussion.

**5. Generalized similarity law for supersonic-hypersonic flow.** When $n$ is negative, the most important non-linear term is the second term in the second square bracket in (12). The flows for which this term is retained we call 'generalized hypersonic flows'. In order to have coefficients in (12) which are independent of Mach number and thickness ratio we must have

$$M^2 \tau^{n'} = \text{constant} = K_4^2 (n' \to 2 \text{ as } M \to \infty) \tag{27}$$

in addition to (14), (16) and (18).

Consideration of (27) and (14) as $M \to \infty$ shows that $n \to -1$ and $K_4 \to K_1$. But $K_4$ and $K_1$ are constants; hence $K_4 = K_1$. Elimination of $\tau$ yields finally

$$n = -\frac{\log \{(M^2 - 1)/K_1^2\}}{\log \{M^4/K_1^2(M^2 - 1)\}} . \tag{28}$$

The variation of $n$ with $M$ is shown for three values of $K_1$ in Figure 2. As before the reliability of the approximations made falls off as $n \to 0$.

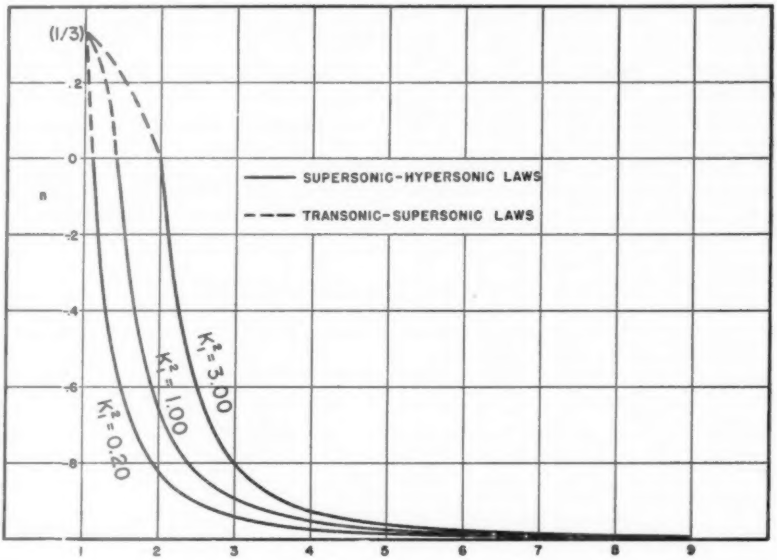


FIG. 2.

6. **Conclusions.** Both the differential equation for the velocity potential and the boundary conditions for the steady irrotational supersonic flow over a thin wing of finite aspect ratio $A$ can be put into a non-dimensional form containing two parameters only, when only the most important of the non-linear terms are retained. The parameters are $K_1 = (M^2 - 1)^{1/2}/\tau^n$ and $K_2 = A(M^2 - 1)^{1/2}$, where $M$ is the Mach number, $\tau$ the ratio of maximum thickness to chord and $n$ is a function of any two of $M$, $\tau$ and $K_1$. The similarity laws thus hold for such profiles and Mach numbers as correspond to the same parameters $K_1$, $K_2$. There are, however, two important restrictions. The most important non-linear term being different in the two cases $n > 0$, $n < 0$, the line $n = 0$ divides the $n,M$-plane into two fundamentally different regions; it is therefore not possible to use a result which corresponds to positive $n$ to deduce a flow which corresponds to negative $n$, or vice-versa. There is a region around $n = 0$ in which neglected and included terms are of approximately the same order of magnitude, so that the reliability falls off appreciably near $n = 0$. Further, the use of the rules should not be attempted for large variations of Mach number, e.g. from a supersonic flow to a genuinely hypersonic flow, as the latter requires the inclusion of a term which is neglected in the present discussion.

For two-dimensional flow $A \to \infty$; the similarity laws then contain only one parameter $K_1$.

For flow over a three-dimensional slender body, including the axially symmetrical flow, $A \to 0$; here also the similarity laws contain only the parameter $K_1$.

In practical cases, $K_1$ is usually larger than one for the transonic-supersonic region and smaller than one for the supersonic-hypersonic region, when $\tau$ is small.

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# LOWER AND UPPER BOUNDS TO THE ULTIMATE LOADS OF BUCKLED REDUNDANT TRUSSES*

BY

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**Synopsis.** In the buckled state, statically indeterminate, rigid-jointed trusses support loads which are generally in excess of those corresponding to initial instability. As buckling proceeds, the loads usually approach limiting values, called "ultimate loads". Two theorems are derived establishing lower and upper bounds to the ultimate loads. Elastic behavior is assumed throughout.

**Introduction.** If the rigid joints of a truss are subjected to a set of external loads, whose values are fixed except for a common multiplier $\lambda_0$, the equations of equilibrium of the $i$-th joint are of the type[1]

$$\sum_j S_{ij}\begin{Bmatrix}\cos\phi_{ij}\\ \sin\phi_{ij}\end{Bmatrix} + \lambda_0\begin{Bmatrix}X_i\\ Y_i\end{Bmatrix} = 0 \qquad (i = 1, 2, \cdots n) \tag{1a}$$

$$\sum_j M_{ij} = 0 \qquad (i = 1, 2, \cdots n) \tag{1b}$$

where $S_{ij}$ and $M_{ij}$ are, respectively, the axial force and bending moment in bar $(i, j)$ at joint $i$, $\phi_{ij}$ is the inclination of bar $(i, j)$, $X_i$ and $Y_i$ are external force components, and where the summation extends over all the bars adjoining $i$. If now all bars are composed of an elastic material and the elementary theory of beams in bending under axial forces is assumed to hold, the joint moments $M_{ij}$ can be expressed as linear functions of the joint rotations $\theta_i$ [2]. Substitution of these expressions into Eqs. (1b) leads to a set of linear, homogeneous equations in the joint rotations of the type

$$\sum_{j=1}^{n} a_{ij}\theta_j = 0, \qquad (i = 1, 2, \cdots n) \tag{2}$$

where the $a_{ij}$ form a symmetric matrix whose components are transcendental functions of the physical characteristics of the bars, and of the axial forces. The equilibrium of the structure is stable only if the matrix is positive definite; a necessary condition of neutral equilibrium is given by

$$f(\lambda_0) \equiv |\, a_{ij} \,| = 0. \tag{3}$$

The lowest positive real root of Eq. (3) is the desired "critical" multiplier $\lambda_0$ at which buckling occurs.

By definition, the bar forces $S_{ij}$ are uniquely determined by Eqs. (1a) for given $\lambda_0$ in a statically determinate truss. If, on the other hand, the truss be of $m$-th degree of redundancy relative to its axial force distribution, the most general expression for the bar forces is of the form

$$S_{ij} = \sum_{\xi=0}^{m} \lambda_\xi S_{ij}^{(\xi)}; \tag{4}$$

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[1]The effect of the linear displacements is ignored in Eqs. (1) and (2). This is in conformity with the experimental results given in Ref. [1] (see the bibliography at the end of the paper).

in Eqs. (4), the force systems $S_{ij}^{(r)}$ are restricted by the set of equilibrium equations

$$\sum_j S_{ij}^{(r)} \begin{Bmatrix} \cos \phi_{ij} \\ \sin \phi_{ij} \end{Bmatrix} + \delta_{0r} \begin{Bmatrix} X_i \\ Y_i \end{Bmatrix} = 0, \qquad (r = 0, 1, \cdots m; i = 1, 2, \cdots n) \tag{5}$$

where $\delta_{pq}$ is the Kronecker Delta.

It is convenient to restrict the force systems $S_{ij}^{(r)}$ further by the requirement (which can always be fulfilled) that

$$\sum_k \frac{S_k^{(0)} S_k^{(r)} L_k}{E_k A_k} = p\delta_{0r} , \qquad (r = 0, 1, 2, \cdots m) \tag{6a}$$

$$\sum_k \frac{S_k^{(r)} S_k^{(s)} L_k}{E_k A_k} = \delta_{rs} , \qquad (r, s = 1, 2, \cdots m) \tag{6b}$$

where the summation extends over all the members of the truss.[2] Eqs. (6a) indicate that the force system $S_{ij}^{(0)}$ is the actual force system, i.e. the one corresponding to minimum strain energy, in the unprestressed, unbuckled truss for $\lambda_0 = 1$.

The axial forces in the buckled state are further restricted by the "compatibility condition" that

$$\delta L_k = (u_j - u_i) \cos \phi_{ij} + (v_j - v_i) \sin \phi_{ij} , \tag{7}$$

in which $\delta L_k$ is the change in the distance between joints $i$ and $j$ relative to the unloaded state and $(u, v)$ are the cartesian components of the linear displacement vectors of the joints. If both sides of Eqs. (7) be multiplied by $S_k^{(r)}$ $(r = 1, 2, \cdots m)$ and summed up over all the members of the truss, we are led, after rearranging the summations and in view of Eqs. (5), to the system of "virtual work" equations

$$\sum_k S_k^{(r)} \delta L_k = 0. \qquad (r = 1, 2, \cdots m) \tag{8}$$

For the sake of generality it will be assumed that the truss has been prestressed, with the initial force system expressed by

$$S_k^* = \sum_{\xi=1}^{m} \lambda_\xi^* S_k^{(\xi)}. \tag{9}$$

On the other hand, $\delta L_k$ can be written in the form

$$\delta L_k = \frac{(S_k - S_k^*) L_k}{E_k A_k} - \delta_k , \tag{10}$$

where $\delta_k$ represents the shortening of the chord length of the $k$-th member corresponding to its curvature in bending. If we substitute Eqs. (10) in (8) and consider Eqs. (6) and (9), we arrive at the following set of equations for any bent state:

$$\lambda_r - \lambda_r^* = \sum_k S_k^{(r)} \delta_k . \qquad (r = 1, 2, \cdots m) \tag{11a}$$

It has been shown elsewhere [3] that, for an actual buckling mode,

$$\sum_k S_k^{(r)} \delta_k = \tfrac{1}{2} \sum_i \sum_j a_{ij,r} \theta_i \theta_j = \mu f_{,r} \qquad (r = 0, 1, 2, \cdots m) \tag{11b}$$

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[2] $S_k$, $L_k$, $E_k$, $A_k$ are the axial force, length, modulus of elasticity and cross-sectional area, respectively, of the $k$-th bar.

in which a subscript, preceded by a comma, designates the partial derivative with respect to the corresponding parameter $\lambda$, or $f_{,r} \equiv \partial f/\partial\lambda_r$. Thus, during buckling,

$$\lambda_r - \lambda_r^* = \mu f_{,r}, \qquad (r = 1, 2, \cdots m) \tag{12a}$$

where $\mu > 0$ is a measure of the extent to which buckling has proceeded and is expressed by

$$\mu = \frac{1}{2} \frac{\sum_{i=1}^{n} \theta_i^2}{\sum_{i=1}^{n} A_{ii}}, \tag{12b}$$

in which $A_{ii}$ is the cofactor of $a_{ii}$ in Eq. (3). It follows immediately from Eqs. (3) and (12) that, as buckling commences, the load $\lambda_0$ will generally increase, and never decrease. Considering all parameters $\lambda_r$ $(r = 0, 1, 2, \cdots m)$ to be functions of $\mu$, Eq. (3) can be written in the form

$$F(\mu) \equiv f[\lambda_0(\mu), \lambda_1(\mu), \cdots \lambda_m(\mu)] \equiv 0 \tag{13}$$

for all values of $\mu$. Differentiating Eq. (13) with respect to $\mu$, we obtain the identity

$$\frac{dF}{d\mu} \equiv f_{,0} \frac{d\lambda_0}{d\mu} + \sum_{\xi=1}^{m} f_{,\xi} \frac{d\lambda_\xi}{d\mu} \equiv 0. \tag{14}$$

On the other hand, by dividing Eq. (12a) by $\mu$ and letting $\mu$ approach zero, we are led to

$$\frac{d\lambda_r}{d\mu}(\mu = 0) = f_{,r}(\mu = 0) \qquad (r = 1, 2, \cdots m) \tag{15}$$

It follows, from Eqs. (14) and (15), that as the truss is on the verge of buckling

$$\frac{d\lambda_0}{d\mu} = -(f_{,0})^{-1} \sum_{\xi=1}^{m} (f_{,\xi})^2, \tag{16}$$

where all the terms in Eq. (16) are to be evaluated at $\mu = 0$.

It can be shown that $f_{,0}$ is negative. In fact, if $\lambda_0^1$ be the smallest positive buckling parameter, i.e. the one corresponding to neutral equilibrium, the positive definiteness of the matrix $[a_{ij}]$ implies $f(\lambda_0) > 0$ for $\lambda_0 < \lambda_0^1$. Since, by Eq. (3), $f(\lambda_0^1) = 0$, and if multiple roots be excluded, it follows that

$$f_{,0}(\lambda_0 = \lambda_0^1) < 0. \tag{17}$$

An inspection of Eqs. (16) and (17) shows that

$$\frac{d\lambda_0}{d\mu}(\mu = 0) \geq 0, \qquad (18)^3$$

where the equality sign applies only to the special case

$$f_{,r}(\lambda_r = \lambda_r^*) = 0. \qquad (r = 1, 2, \cdots m)$$

It follows further from Eqs. (12) that for increasing $\mu$, and provided the $\lambda_r$ all stay

[3] Actually, this relationship applies to all values of $\mu$, as can be demonstrated by means of energy considerations.

finite,[4] the external and internal forces approach a state governed by the equations

$$\begin{aligned} f(\lambda_0^u, \lambda_1^u, \cdots \lambda_m^u) &= 0, \\ f_{,r}(\lambda_0^u, \lambda_1^u, \cdots \lambda_m^u) &= 0, \qquad (r = 1, 2, \cdots m) \end{aligned} \tag{19}$$

in which the superscript designates limiting, or "ultimate" values of the parameters, with $\lambda_0^u$ referred to as the "ultimate load".

The determination of $\lambda_0^u$ is of interest to the engineer since it represents the ultimate carrying capacity of the truss. Since, however, the solution of Eqs. (19) presents formidable numerical obstacles, it is easier to estimate the value of $\lambda_0^u$ by means of two theorems establishing lower and upper bounds to the ultimate load.

**A stability criterion.** Before proceeding to these two theorems, it is convenient to establish a stability criterion for the unbuckled truss, which is to be used in the later proof. This criterion is based on the assumption that, for a given system of external forces identified by the load parameter $\lambda_0$, the potential energy corresponding to the unbuckled state is less than that of any neighboring, "geometrically consistent" bent state if the equilibrium of the truss is to be stable, or

$$V^* < V. \tag{20}$$

In what follows, a geometrically consistent bent state is defined as one in which the deflection curves of the bars are of sufficient degree of smoothness to make the discussion meaningful, and satisfy the geometric boundary conditions of continuity at the joints. Geometric consistency also implies the satisfaction of Eqs. (7) and (10); it is finally assumed that the force equations (4), (5), and (6), and therefore also Eqs. (8) and (11a), are valid.

With these definitions, the potential energy

$$V = U - W = \frac{1}{2} \sum_k \frac{S_k^2 L_k}{E_k A_k} + U_B - \lambda_0 \sum_i (X_i u_i + Y_i v_i),$$

in which $U_B$ is the strain energy associated with the bending of the truss bars, can be expressed by Eqs. (4), (5), (6) and (7) in the form

$$V = \frac{1}{2} p\lambda_0^2 + \frac{1}{2} \sum_{\xi=1}^{m} \lambda_\xi^2 + U_B - \sum_k S_k \delta L_k \ ;$$

in view of Eqs. (8), (9), and (10), this reduces to

$$V = -\frac{1}{2} p\lambda_0^2 + \frac{1}{2} \sum_{\xi=1}^{m} \lambda_\xi^2 + U_B + \lambda_0 \sum_k S_k^{(0)} \delta_k \ .$$

Similarly, the potential energy in the unbent state is given by

$$V^* = -\frac{1}{2} p\lambda_0^2 + \frac{1}{2} \sum_{\xi=1}^{m} \lambda_\xi^{*2}.$$

Thus, criterion (20) now takes the form

$$V - V^* = \frac{1}{2} \sum_{\xi=1}^{m} (\lambda_\xi - \lambda_\xi^*)(\lambda_\xi + \lambda_\xi^*) + U_B + \lambda_0 \sum S_k^{(0)} \delta_k \geq 0$$

[4]For a discussion of this question see Ref. [3].

which, by Eqs. (11a) and by considering only states in the neighborhood of the unbuckled state, is finally expressed as

$$V - V^* = \sum_k S_k \delta_k + U_B \geq 0, \tag{21}$$

where the equality sign applies to the case of neutral equilibrium, with the bent state represented by the first buckling mode. The stability criterion (21) can also be demonstrated directly. In fact, if the arbitrary set of deflection curves $y_k(x)$ be developed in a Fourier series

$$y_k(x) = \sum_{r=1}^{\infty} c_r \eta_k^r(x).$$

in which $\eta_k^r(x)$ represents the normalized buckling mode associated with the positive load parameter $\lambda_0^r$, it can be shown, by a series of partial integrations and by virtue of the orthogonality of the normal modes, that

$$\sum_k S_k \delta_k + U_B = \sum_{r=1}^{\infty} c_r^2(\lambda_0^r - \lambda_0).$$

The right side of this equation is positive definite if $\lambda_0 < \lambda_0^1$, i.e. when the load parameter is smaller than the smallest buckling load.

**Lower bounds to the ultimate load.** After these preliminary remarks, we now proceed to state the first theorem. In what follows, we shall call a load parameter $\lambda_0 > 0$ "critical" if there exists a set of prestressing parameters $\lambda_r^*$ $(r = 1, 2 \cdots m)$ such that the truss is in neutral equilibrium with its force system defined by $(\lambda_0, \lambda_1^*, \cdots \lambda_m^*)$. With this definition, which is analogous to that of a "statically admissible multiplier" in rigid-plastic limit design [4], we state

THEOREM I: *The ultimate load is the largest of all critical loads.*

From this theorem we deduce the following

COROLLARY: If an arbitrary internal force system, and an external load system identified by $\lambda_0'$, satisfy the force equations of equilibrium, and if the truss, with its forces so defined, is in stable equilibrium, then

$$\lambda_0^u > \lambda_0'. \tag{22}$$

It is clear that this corollary establishes an easily calculable lower bound to the ultimate load since the stability of a truss for a given force system can be determined by a number of stability criteria [5, 6].

PROOF: It can readily be verified that if the prestressing parameters be so selected that $\lambda_r^* = \lambda_r^u$ $(r = 1, 2 \cdots m)$, Eqs. (3) and (12a) are identically satisfied by

$$\lambda_r \equiv \lambda_r^u \qquad (r = 0, 1, 2, \cdots m)$$

for all values of $\mu$. In other words, in this "special case", the truss buckles similarly to a statically determinate truss under constant external loads and internal forces.[5]

[5]It has been shown previously [7] that this case corresponds to a stationary value of the buckling load. See also Eq. (18).

Inspection of Eq. (11a) shows that the buckling mode of the special case is such as to satisfy

$$\sum_k S_k^{(r)} \delta_k = 0. \qquad (r = 1, 2, \cdots m) \tag{23}$$

If now the stability criterion (21) be applied to a truss in stable equilibrium under an external load $\lambda_0'$, and if the arbitrary deflection curves be so selected as to coincide with the buckling mode of the "special case", it follows from Eqs. (23) that

$$V - V^* = \lambda_0' \sum_k S_k^{(0)} \delta_k + U_B > 0.$$

Since, also by Eqs. (21) and (23),

$$\lambda_0'' \sum_k S_k^{(0)} \delta_k + U_B = 0,$$

and since further $U_B$ is positive definite, it follows that

$$\lambda_0'' > \lambda_0' . \tag{22}$$

**Upper bounds to the ultimate load.** Before proceeding to the discussion of upper bounds to the ultimate load, it is useful to state the following

LEMMA I: For a given set of prestressing parameters, the value of the critical load of a truss is not reduced by increasing the moment of inertia of a bar.[6]

PROOF: As stated before, a truss is in stable equilibrium only if the matrix of the coefficients $a_{ij}$ is positive definite, or

$$Q = \tfrac{1}{2} \sum_i \sum_j a_{ij} \theta_i \theta_j \geq 0 \tag{24}$$

where the equality sign applies only to the trivial case $\theta_i = 0$ $(i = 1, 2 \cdots n)$. By rearranging the terms in the double sum, this can be expressed [8] in the form

$$Q = \tfrac{1}{8} \sum_k S_k L_k [(\epsilon_k \coth \epsilon_k - 1)^{-1} (\theta_k + \theta_k')^2 + (\epsilon_k)^{-1} (\theta_k - \theta_k')^2 \coth \epsilon_k], \tag{25a}$$

where $\epsilon_k$ is defined by

$$\epsilon_k^2 = \frac{S_k L_k^2}{4 E_k I_k} > -\pi^2, \tag{25b}$$

and where $\theta_k$ and $\theta_k'$ are, respectively, the rotations of the two joints connected by the $k$-th bar of moment of inertia $I_k$.

Let us now assume that a truss $T$ is in stable equilibrium for a given set of internal and external forces, and consider the equilibrium, under the same force system, of a truss $T'$ which is identical with $T$ as to its geometric and elastic properties, except that the moment of inertia of its $m$-th bar has been increased, i.e.

$$I_m' > I_m .$$

If this bar be a tension member, it follows from Eq. (25b) that

$$(\epsilon_m')^2 < \epsilon_m^2$$

[6]This lemma, which has immediate physical appeal, is analogous to a similar principle of Rayleigh dealing with the natural frequencies of vibration of elastic systems.

and, by inspection of Eq. (25a), that the value of the $m$-th term in the series has not decreased. The same can readily be demonstrated for compression members, which correspond to imaginary values of $\epsilon$, and for unstressed members. Consequently, for any given set of joint rotations,

$$Q' \geq Q;$$

in view of Eq. (24), the truss $T'$ is therefore also in stable equilibrium.

As a next step, it is easy to establish the following

LEMMA II: The value of the ultimate load of a truss is not lowered by increasing the moment of inertia of a bar.

PROOF: This is a direct consequence of the previous lemma and of Theorem I. In fact, if the prestressing parameters $\lambda_r^*$ $(r = 1, 2 \cdots m)$ be so selected as to make the critical load $\lambda_0$ equal to the ultimate load $\lambda_0^u$ for truss $T$, it follows from Lemma I that, for the same $\lambda_r^*$, the critical load $\lambda_0'$ of truss $T'$ satisfies the relationship

$$\lambda_0' \geq \lambda_0 = \lambda_0^u .$$

On the other hand, it follows from Theorem I that the ultimate load $\lambda_0'^u$ of truss $T'$ is the largest of all critical loads of $T'$, or

$$\lambda_0'^u \geq \lambda_0'$$

which, in view of the foregoing, implies

$$\lambda_0'^u \geq \lambda_0^u . \tag{26}$$

An upper bound to the ultimate load can now be established by means of the fact that a truss of $m$-th degree of redundancy can be converted into a rigid-link mechanism of one degree of freedom by considering it pin-jointed and by removing $l$ members, where $l \leq m + 1$. In general, it will be possible[7] to select these $l$ members in such a way that the mechanism so created is capable of joint velocities corresponding to a shortening of all the eliminated bars.

If now these $l$ members be assigned compressive forces

$$S_k = -4\pi^2 E_k I_k (L_k)^{-2} \qquad (k = 1, 2, \cdots l) \tag{27}$$

and if the remaining bar forces be so chosen as to satisfy the force equations of equilibrium (2a) for $\lambda_0 = \lambda_0'' > 0$, then, by

THEOREM II:

$$\lambda_0^u \leq \lambda_0'' . \tag{28}$$

In other words, the load parameter found by the method described above represents an upper bound to the ultimate load.

The proof of this follows from the second lemma. In fact, since Eq. (27) defines the buckling load of a column which is fixed at both ends, it is apparent that $\lambda_0''$ can be interpreted as the ultimate load of a truss which is identical with the truss under

[7]Otherwise, the truss does not exhibit an ultimate load, i.e., the load increases indefinitely during buckling.

consideration except that all bars other than the selected $l$ bars described above have infinite moment of inertia. Thus the second lemma implies the relationship (28).

It should be pointed out that the practical usefulness of the upper bound defined by Theorem II is sharply limited. A more detailed discussion of this question can be found elsewhere [3]; suffice it to state here that the theorem is much less broad in scope than a similar one establishing upper bounds to the collapse load of a plastic-rigid frame. [4] This is due to the fact that it has not been possible to construct an infinity of readily calculable "kinematically admissible" collapse modes by any simple process analogous to the insertion of a sufficient number of yield hinges. Fortunately, except for the purpose of estimating the degree of accuracy attained, the interest of the engineer is focused on the lower bound only, which can be made to approach the exact value of the collapse load as closely as desired.

**Conclusion.** Two principles have been derived which establish lower and upper bounds to the ultimate load sustained by a buckled redundant truss. Of these, the lower bound permits an approach to the exact value from below within any desired degree of accuracy.

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# ON SAINT-VENANT'S PRINCIPLE*

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**Introduction.** The principle bearing his name was introduced by Saint-Venant [1][1] in connection with, and with limitation to, the problem of extension, torsion, and flexure of prismatic and cylindrical bodies. The first universal statement of the principle is apparently due to Boussinesq [2], and reads:[2] "An equilibrated system of external forces applied to an elastic body, all of the points of application lying within a given sphere, produces deformations of negligible magnitude at distances from the sphere which are sufficiently large compared to its radius." Love [3] writes:[3] "According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part."

As pointed out by v. Mises [4], the foregoing statements are in need of clarification[4] since the forces applied to a body at rest must be in equilibrium in any event. Only when the body extends to infinity, and provided we require the tractions at infinity to vanish suitably, is it meaningful to speak of the strains "produced" by a non-equilibrated system of forces applied to a bounded part of its surface. Moreover, in this instance, the strains produced by a given loading are arbitrarily small at points sufficiently far removed from the region of load application, regardless of whether or not the loading is self-equilibrated.[5] On the other hand, the stresses and strains at a fixed point of an elastic body, in the absence of body forces, may be made arbitrarily large or small by choosing the magnitude of the loads sufficiently large or small. These observations further confirm the need for clarification.

What is meant by the statements cited may roughly be expressed as follows:[6] if the forces acting on an elastic body are confined to several distinct portions of its surface, each lying within a sphere of radius $\epsilon$, then the stresses and strains at a fixed interior point of the body are of a smaller order of magnitude in $\epsilon$ as $\epsilon \to 0$ when the forces on each of the portions are in equilibrium than when they are not. In this comparison we must evidently assume that the forces remain bounded as $\epsilon \to 0$. The analogous interpretation for distributed surface tractions is immediate.

It should be noted that such an interpretation is implied in the usual applications of Saint-Venant's principle. Moreover, that this is what Boussinesq had in mind is apparent from his efforts to justify the principle. With this objective, Boussinesq [2] considered

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[1]Numbers in brackets refer to the bibliography at the end of this paper.

[2]See [2], p. 298. ("Des forces extérieures, qui se font équilibre sur un solide élastique et dont les points d'application se trouvent tous à l'intérieur d'une sphère donnée, ne produisent pas de déformations sensibles à des distances de cette sphère qui sont d'une certaine grandeur par rapport à son rayon.")

[3]See [3], p. 132.

[4]See also, for example, Biezeno and Grammel [5], where the traditional statement of the principle is discussed in detail.

[5]See the general solution to the problem of a semi-infinite medium bounded by a plane, [3], art. 166.

[6]This interpretation follows v. Mises [4].

a semi-infinite body under concentrated loads acting perpendicular to its plane boundary. He showed that if the points of application of the loads lie within a sphere of radius $\epsilon$, the stresses at a fixed interior point of the body are of the order of magnitude $\epsilon$ provided the resultant force is zero, and of the order $\epsilon^2$ in case the resultant moment also vanishes. Various energy arguments have since been advanced in support of the principle.[7]

In 1945 v. Mises [4], in his illuminating paper on this subject, showed with the aid of two specific examples that the usual statements of the principle, when properly clarified, cannot be valid without qualifications. The two examples chosen by v. Mises are the three-dimensional problem of the half-space and the plane problem of the circular disk, each under concentrated surface loads.[8] On the basis of these examples v. Mises proposed an amended principle.

It is the purpose of this paper to supply a general proof of Saint-Venant's principle as modified by v. Mises. The argument is carried on for the case of piecewise continuous tractions and is later extended to concentrated forces; it applies to finite and infinite domains of arbitrary connectivity.[9]

**The dilatation formula of Betti.** As a preliminary to the proof, we recall here a formula due to Betti,[10] which is a consequence of Betti's reciprocal theorem. Let $D$ be a regular[11] (not necessarily simply connected) region occupied by an elastic medium, and let $B$ be the boundary of $D$ (Figure 1). Furthermore,[12] let $\mathbf{u}$, $e_{ij}$, and $\tau_{ij}$ be a displacement field, a field of strain, and a field of stress which within $D$ satisfy the fundamental field equations of the linear theory of elasticity in the absence of body forces. If $\tau_{ij}$ gives rise to piecewise continuous surface tractions $\mathbf{T} = [X_1, X_2, X_3]$ on $B$, then the dilatation $\Delta^Q = e_{ii}^Q$ at a fixed interior point $Q(\xi_1, \xi_2, \xi_3)$ of $D$ is given by,[13]

$$c\Delta^Q = \int_B \mathbf{T} \cdot \mathbf{g} \, dB, \qquad c = \frac{8\pi(1-\nu)\mu}{1-2\nu}, \tag{1}$$

where $\mu$ and $\nu$ are the shear modulus and Poisson's ratio, respectively. In (1), $\mathbf{g}$ is a displacement field which is defined as follows:

$$\mathbf{g} = \mathbf{g}' + \mathbf{g}'', \qquad \mathbf{g}' = -\operatorname{grad} R^{-1}, \qquad R = |\mathbf{R}|, \tag{2}$$

where $\mathbf{R}$ is the position vector with respect to $Q$ of a point $P(x_1, x_2, x_3)$ of $D$ (Figure 1). Moreover, $\mathbf{g}''$ is that displacement field which satisfies the equilibrium equations within $D$ and gives rise to surface tractions on $B$ which are equal and opposite to those associated with $\mathbf{g}'$. Thus $\mathbf{g}$ is characterized by the requirements that (a) it satisfy the equilibrium conditions inside $D$ with the exception of the point $Q$ where it must have the singularity appropriate to a center of dilatation, and (b) its associated surface tractions vanish on $B$. We emphasize, for future reference, that $\mathbf{g}$ is an analytic function of position on any analytic portion of $B$. Betti's formula remains valid if $D$ is not bounded, provided that $R^2\tau_{ij} \to 0$ as $R \to \infty$.

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[7]See References [6] to [11] and [18].

[8]More recently, Erim [12], applied v. Mises' analysis to the half-plane under concentrated loads.

[9]The restriction that the region be simply connected turns out to be unessential.

[10]See [3], p. 235.

[11]The term "regular region" is used in the sense of Kellogg [13], pp. 113, 217.

[12]Throughout this paper letters in boldface designate vectors; the subscripts $i$, $j$ assume the values 1, 2, 3, and the usual summation convention is employed.

[13]The symbols $\cdot$ and $\times$ designate scalar and vector multiplication of two vectors, respectively.

The vector field **g** plays a role analogous to that of Green's function in potential theory. Equation (1) reduces the determination of the dilatation in the second boundary-value problem (surface tractions prescribed) to the determination of $\mathbf{g}''$. We note that the function **g** is completely characterized by the shape of $D$ and the location of $Q$, and is independent of the surface tractions **T**. This observation provides the key to the subsequent proof of the modified Saint-Venant principle in which we are confronted with the task of comparing the effects of certain changes in the loading upon the de-

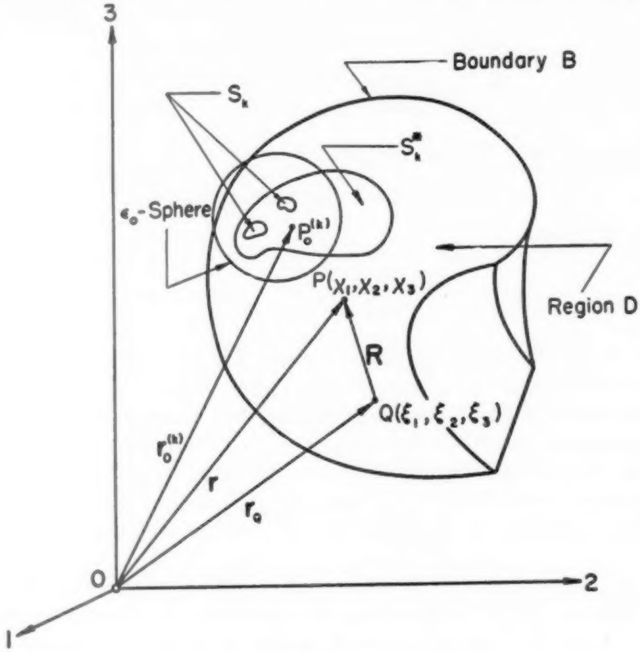


FIG. 1.

formations and stresses at an interior point of the body. Before turning to our main objective, we consider two specific applications of Betti's formula.

**The sphere and the half-space as examples.** Let $D$ be a sphere with radius $a$ and let $Q$ be its center. Here, trivially,

$$\mathbf{g}'' = \frac{2(1-2\nu)\mathbf{R}}{(1+\nu)a^3}, \tag{3}$$

and by (1), (2), (3),

$$\Delta^Q = \frac{3(1-2\nu)}{8\pi(1+\nu)\mu a^3}\int_B \mathbf{T}\cdot\mathbf{R}\,dB. \tag{4}$$

If the loading, in particular, consists of two equal and opposite concentrated forces, each of magnitude $L$, applied at the endpoints of a diameter and directed toward $Q$, a trivial limit process applied to (4) at once yields,

$$\Delta^Q = -\frac{3(1-2\nu)L}{4\pi(1+\nu)\mu a^2}\,. \tag{5}$$

This formula was derived by Synge [14], by Weber [15], as well as by F. Rosenthal and the present author [16], in each case by entirely different means.

Next, let $D$ be the half-space $x_3 \geq 0$ and $B$ the plane $x_3 = 0$. In this special instance a closed representation of $\mathbf{g}$ is available[14] corresponding to any interior point $Q(\xi_1, \xi_2, \xi_3)$. Indeed, let

$$\phi = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 + \xi_3)^2]^{-1/2}, \tag{6}$$

so that $\phi$ is the reciprocal of the distance between $P(x_1, x_2, x_3)$ and the mirror image of $Q$ in the plane $x_3 = 0$. Then, as is readily verified,

$$\mathbf{g}'' = -(3 - 4\nu)\left[\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_3}\right] - 2x_3 \operatorname{grad} \frac{\partial \phi}{\partial x_3} \tag{7}$$

and, according to (2), (6), (7), on the plane $x_3 = 0$,

$$\mathbf{g} = -4(1 - \nu)[\operatorname{grad} R^{-1}]_{(x_3=0)} . \tag{8}$$

Equations (1), (8) imply,

$$\Delta^Q = -\frac{(1 - 2\nu)}{2\pi\mu} \int_B \mathbf{T} \cdot \operatorname{grad} R^{-1} \, dB. \tag{9}$$

An elementary limit process applied to (9) yields the formula cited by v. Mises[15] for the dilatation at $Q$ induced by a concentrated load acting at the origin $x_i = 0$.

**Proof of the modified Saint-Venant principle.** Let $D$, with the boundary $B$, again be a regular region[16] of space of arbitrary connectivity. Let $S_k(k = 1, 2, \cdots m)$ be $m$ non-intersecting closed subregions of $B$ which lie within neighborhoods of $m$ distinct points $P_0^{(k)}$ (position vectors $\mathbf{r}_0^{(k)}$) of $B$, each $S_k$, $P_0^{(k)}$ being wholly contained within a sphere of radius $\epsilon_0$ (Figure 1). We note that $S_k$ need not be simply connected or even connected.

Let $\mathbf{u}$, $e_{ij}$, $\tau_{ij}$ be a solution in $D$ of the field equations of elasticity theory, the body forces being absent, which corresponds to piecewise continuous surface tractions $\mathbf{T} = [X_1, X_2, X_3]$ on $B$.[17] Moreover, let $\mathbf{T}$ vanish on $B$ with the exception of the subregions $S_k$.

According to Betti's formula (1),

$$\Delta^Q = \sum_{k=1}^{m} \Delta^Q[S_k], \qquad c\Delta^Q[S_k] = \int_{S_k} \mathbf{g} \cdot \mathbf{T} \, dB \tag{10}$$

where $\Delta^Q[S_k]$ represents the "contribution" from the tractions on $S_k$ to the dilatation $\Delta^Q$ at a fixed interior point $Q$ of $D$. We observe that a $\Delta^Q[S_k]$ possesses individual physical significance only if the tractions on $S_k$ are self-equilibrated, unless $D$ extends to infinity. Furthermore, if $D$ is bounded, either the tractions on each $S_k$ are self-equilibrated, or there are at least two $S_k$ for which this is not true.

We now examine $\Delta^Q[S_k]$ and, for the sake of convenience, henceforth write $S$, $P_0$, $\mathbf{r}_0$

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[14] This result is apparently due to Cerruti. See [3], p. 239.

[15] See Equation (4) of [4].

[16] See Footnote 11. Again $D$ need not be bounded.

[17] If $D$ is not bounded, we require $r^2\tau_{ij} \to 0$ as $r \to \infty$, where $r = |\mathbf{r}|$, and $\mathbf{r}$ is the position vector of a point $P$ of $D$.

in place of $S_k$, $P_0^{(k)}$, $\mathbf{r}_0^{(k)}$. Let $S$ and $P_0$ be contained in an open simply connected subregion $S^*$ of $B$ (Figure 1) which admits a parametrization of the form,[18]

$$\mathbf{r} = \mathbf{r}(\alpha, \beta), \qquad \mathbf{r}_\alpha \times \mathbf{r}_\beta \neq 0, \qquad (\alpha, \beta) \quad \text{in} \quad \Sigma^* \tag{11}$$

where $\mathbf{r} = [x_1, x_2, x_3]$ here is the position vector of a point $P$ of $S^*$, $\Sigma^*$ is an open simply connected region of the $(\alpha,\beta)$-plane, and $\mathbf{r}(\alpha, \beta)$ is assumed to be at least twice continuously differentiable in $\Sigma^*$. Thus $B$ is assumed to have finite and continuous curvatures in $S^*$. The mapping (11) defines a regular curvilinear coordinate net on $S^*$. It is convenient to require that

$$\mathbf{r}_0 = \mathbf{r}(0, 0). \tag{12}$$

Finally, suppose that the closed subregion $\Sigma$ of $\Sigma^*$ is the antecedent in the $(\alpha,\beta)$-plane of the subregion $S$ of $S^*$.

From (10), by virtue of the regularity of $\mathbf{g}$ on $B$, we have, on expanding $\mathbf{g}(\alpha, \beta)$ in a Taylor series (possibly with a remainder term) at (0, 0),[19]

$$c\Delta^Q[S] = \mathbf{g}^0 \cdot \int_S \mathbf{T}\, d\sigma + \mathbf{g}_\alpha^0 \cdot \int_S \mathbf{T}\alpha\, d\sigma + \mathbf{g}_\beta^0 \cdot \int_S \mathbf{T}\beta\, d\sigma + \cdots \tag{13}$$

where,

$$\int_S d\sigma = \int_\Sigma |\, \mathbf{r}_\alpha \times \mathbf{r}_\beta \,|\, d\alpha\, d\beta. \tag{14}$$

Before drawing any general conclusions, let us apply (13) to the example of the half-space. Here $D$ is the region $x_3 \geq 0$ and $B$ the plane $x_3 = 0$. We may choose the origin $x_i = 0$ at $P_0$, employ $B$ as $S^*$, and adopt the parametrization,

$$\alpha = x_1, \qquad \beta = x_2. \tag{15}$$

By (8), in this instance,

$$\left.\begin{aligned} \mathbf{g}(\alpha, \beta) &= -\frac{4(1-\nu)}{R^3}[\xi_1 - \alpha, \xi_2 - \beta, \xi_3] \\ R &= [(\xi_1 - \alpha)^2 + (\xi_2 - \beta)^2 + \xi_3^2]^{1/2} \end{aligned}\right\} \tag{16}$$

Substitution of (16) into (13) yields, except for differences in notation and a constant factor, the expansion derived by v. Mises by other means, provided the integrals in (13) are replaced with the corresponding finite sums.[20]

We now return to (13) and to our main objective, which is to examine the order of magnitude of the dilatation at the fixed interior point $Q$ of $D$ in relation to the size of the region $S$, under various assumptions regarding the tractions on $S$ (e.g., if the tractions on $S$ are or are not self-equilibrated). To this end we first recall the mathematical

[18]The subscripts $\alpha$, $\beta$ denote partial differentiation with respect to the argument indicated. The existence of such a regular parametrization is assured in the small, provided $S^*$ is sufficiently smooth. Note that the embedding regions $S_k^*$, belonging to the various $S_k$, $P_0^{(k)}$, in general, require different parametrizations.

[19]The superscript zero attached to any function of $(\alpha, \beta)$ refers to its value at (0, 0).

[20]See Equation (7) of [4]; this equation gives the mean normal stress rather than the dilatation at $Q$, and applies to concentrated forces. The transition from distributed tractions to concentrated forces will be discussed later. See Equation (29) of this paper.

meaning of the concept of "order of magnitude". If[21]

$$|f(x)/x^p| < M \quad (M \text{ independent of } x) \quad \text{as} \quad x \to 0, \qquad \text{then} \qquad f(x) = O(x^p), \tag{17}$$

that is, $f(x)$ is said to be of the same order of magnitude as $x^p$. It is clear from (17) that the question as to the order of magnitude of $\Delta^Q[S]$ in (13), with respect to the radius $\epsilon_0$ of the sphere enclosing $S$ and $P_0$, has no meaning since $\epsilon_0$ is a number and not a variable.

The foregoing question becomes meaningful, however, if we ask what happens in the limit as the region $S$ is contracted to the fixed point $P_0$ of $B$. To make this idea precise, consider a one-parameter family of closed subregions $S(\epsilon)$ of $S^*$ such that for every $\epsilon$ in $0 < \epsilon \leq \epsilon_0$, $S(\epsilon)$ together with $P_0$ lies within a sphere of radius $\epsilon$, $S(\epsilon_0) = S$, and the maximum diameter $d(\epsilon)$ of $S(\epsilon)$ is a monotone increasing function of $\epsilon$. Next, let $\mathbf{u}(\epsilon)$, $e_{ij}(\epsilon)$, $\tau_{ij}(\epsilon)$ $(0 < \epsilon \leq \epsilon_0)$ be a one-parameter family of solutions in $D$ of the field equations of elasticity theory, the body forces being absent, which satisfies the following conditions: $\mathbf{u}(\epsilon_0) = \mathbf{u}$; $\tau_{ij}(\epsilon)$ gives rise to piecewise continuous surface tractions $\mathbf{T}(\epsilon) = [X_1(\epsilon), X_2(\epsilon), X_3(\epsilon)]$ which vanish on $B$ with the exception[22] of the subregions $S(\epsilon)$; $\mathbf{T}(\epsilon)$ remains bounded as $\epsilon \to 0$.

Writing $\Delta^Q(\epsilon)$ for $\Delta^Q[S(\epsilon)]$, we have from (13), (14),

$$c\Delta^Q(\epsilon) = \mathbf{g}^0 \cdot \int_{S(\epsilon)} \mathbf{T}(\epsilon)\, d\sigma + \mathbf{g}^0_\alpha \cdot \int_{S(\epsilon)} \mathbf{T}(\epsilon)\alpha\, d\sigma + \mathbf{g}^0_\beta \cdot \int_{S(\epsilon)} \mathbf{T}(\epsilon)\beta\, d\sigma + \cdots, \tag{18}$$

$$\int_{S(\epsilon)} d\sigma = \int_{\Sigma(\epsilon)} |\mathbf{r}_\alpha \times \mathbf{r}_\beta|\, d\alpha\, d\beta, \tag{19}$$

where $\Sigma(\epsilon)$ is the antecedent of $S(\epsilon)$ in the $(\alpha,\beta)$-plane. Furthermore, in view of the regularity of the mapping (11), and from (12),

$$\delta(\epsilon) = \max_{\Sigma(\epsilon)} (\alpha^2 + \beta^2)^{1/2} \to 0 \qquad \text{as} \qquad \epsilon \to 0 \qquad \text{and} \qquad \delta = 0(\epsilon). \tag{20}$$

Let

$$\mathbf{F}(\epsilon) = \int_{S(\epsilon)} \mathbf{T}(\epsilon)\, d\sigma, \qquad \mathbf{M}(\epsilon) = \int_{S(\epsilon)} \mathbf{r} \times \mathbf{T}(\epsilon)\, d\sigma \qquad (0 < \epsilon \leq \epsilon_0). \tag{21}$$

Thus, $\mathbf{F}(\epsilon)$ and $\mathbf{M}(\epsilon)$ are the resultant force and the resultant moment in $0(\mathbf{r} = 0)$ of the tractions on $S(\epsilon)$. Equation (18), in conjunction with (19), (20), (21), now yields the following conclusions:

(a) $\mathbf{F}(\epsilon) \neq 0$ $(0 < \epsilon \leq \epsilon_0)$, *in general, implies* $\Delta^Q(\epsilon) = 0(\epsilon^2)$. Exceptions are possible, however. In the example of the sphere, we see from (4) that $\Delta^Q = 0$ whenever the surface tractions are purely tangential, provided $Q$ is the center of the sphere.

(b) $\Delta^Q(\epsilon) = 0(\epsilon^3)$ *or smaller if* $\mathbf{F}(\epsilon) = 0 \quad (0 < \epsilon \leq \epsilon_0)$.

(c) $\Delta^Q(\epsilon) = 0(\epsilon^4)$ *or smaller if*

$$\mathbf{F}(\epsilon) = 0, \qquad \int_{S(\epsilon)} \mathbf{T}(\epsilon)\alpha\, d\sigma = 0, \qquad \int_{S(\epsilon)} \mathbf{T}(\epsilon)\beta\, d\sigma = 0 \qquad (0 < \epsilon \leq \epsilon_0). \tag{22}$$

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[21]This limited form of the definition is sufficient for our purposes.

[22]Recall that $S(\epsilon)$ stands for $S_k(\epsilon)$ $(k = 1, 2, \cdots m)$. The piecewise continuity of $\mathbf{T}$ does not refer to its dependence on $\epsilon$ for which no continuity requirements are imposed.

Within terms of $0(\epsilon^3)$, Equations (22) are equivalent[23] to the 12 scalar conditions,

$$\int_{S(\epsilon)} X_i(\epsilon)\, d\sigma = 0, \qquad \int_{S(\epsilon)} X_i(\epsilon) x_j\, d\sigma = 0 \qquad (0 < \epsilon \leq \epsilon_0). \tag{23}$$

*Hence, if the tractions on* $S(\epsilon)$ *satisfy* (23) *then* $\Delta^Q(\epsilon) = 0(\epsilon^4)$ *or smaller.*

(d) Equations (23) imply,

$$\mathbf{F}(\epsilon) = 0, \qquad \mathbf{M}(\epsilon) = 0 \qquad (0 < \epsilon \leq \epsilon_0), \tag{24}$$

but the converse is not true, in general. Within terms of $0(\epsilon^3)$ the equilibrium conditions (24), in view of (21), (11), are equivalent[23] to

$$\mathbf{F}(\epsilon) = 0, \qquad \mathbf{r}^0_\alpha \times \int_{S(\epsilon)} \mathbf{T}(\epsilon)\alpha\, d\sigma + \mathbf{r}^0_\beta \times \int_{S(\epsilon)} \mathbf{T}(\epsilon)\beta\, d\sigma = 0 \qquad (0 < \epsilon \leq \epsilon_0). \tag{25}$$

Again, these conditions are met if (22) hold, but (22) do not follow from (25). *Thus, if the tractions on* $S(\epsilon)$ *are self-equilibrated* $\Delta^Q(\epsilon) = 0(\epsilon^3)$ *or smaller.* This conclusion contradicts the interpretation of the traditional statement of Saint-Venant's principle cited in the Introduction, according to which the order of magnitude of $\Delta^Q(\epsilon)$ should always be smaller when the tractions on $S(\epsilon)$ are self-equilibrated than when they are not.

(e) Suppose, in particular, the tractions on $S(\epsilon)$ are parallel, so that

$$\mathbf{T}(\alpha, \beta; \epsilon) = \mathbf{k}T(\alpha, \beta; \epsilon) \qquad \text{in} \qquad \Sigma(\epsilon) \qquad (0 < \epsilon \leq \epsilon_0), \tag{26}$$

in which $\mathbf{k}$ is a fixed vector and $T$ a scalar function. Here (23) are satisfied if and only if (24) hold for every choice of $\mathbf{k}$, that is, if and only if the system of parallel tractions remains in equilibrium under an arbitrary change of its direction, the magnitude and sense of the tractions being maintained ("astatic equilibrium"). *Therefore, in the event the tractions on* $S(\epsilon)$ *are parallel and in astatic equilibrium,* $\Delta^Q(\epsilon) = 0(\epsilon^4)$ *or smaller.*

(f) Suppose (26) holds and, in addition,

$$\mathbf{k} \cdot (\mathbf{r}^0_\alpha \times \mathbf{r}^0_\beta) \neq 0, \tag{27}$$

so that $\mathbf{k}$ is not parallel to the tangent plane of $S^*$ at $P_0$. Then (25) imply (22). *Hence, if the tractions on* $S(\epsilon)$ *are parallel to each other, self-equilibrated, and not parallel to a tangent plane of* $S^*$, *then* $\Delta^Q(\epsilon) = 0(\epsilon^4)$ *or smaller.* The analogous conditions for concentrated forces were satisfied in the special example investigated by Boussinesq [2] and described in the Introduction.

We have so far considered only the order of magnitude of the dilatation. According to Lauricella,[24] the strains $e^Q_{ij}$ at an interior point $Q$ of $D$, in the absence of body forces, admit the representation,

$$e^Q_{ij} = \int_B \mathbf{T} \cdot \mathbf{g}_{ij}\, dB, \tag{28}$$

which is analogous to that given by Betti's formula (1) for the dilatation $\Delta^Q$. Here the $\mathbf{g}_{ij}$ are displacement fields which satisfy the equilibrium conditions inside $D$, with the

[23]In case $S^*$ is plane, and for the parametrization (15), the equivalence is exact.

[24]See [3], p. 216.

exception of the point $Q$ where they have certain prescribed singularities,[25] and which give rise to vanishing surface tractions on $B$. Clearly, the $\mathbf{g}_{ij}$ are again regular on any regular portion of $B$, and the previous argument applies without modification. Therefore, the conclusions listed under (a) to (f) remain valid if $\Delta^Q(\epsilon)$ is replaced with $e_{ij}^Q(\epsilon)$, and hence with $\tau_{ij}^Q(\epsilon)$. If a rigid displacement of the whole body is excluded, the conclusions also apply to the displacement $\mathbf{u}^Q(\epsilon)$, as follows from Somigliana's representation of the displacement field.[26] Specific illustrations of these general conclusions were given by v. Mises in [4].[27]

**Extension to the case of concentrated forces. Remarks.** The theorem proved in the preceding section is readily extended to concentrated forces. With reference to (13), let the concentrated forces $\mathbf{T}_n = [X_1^{(n)}, X_2^{(n)}, X_3^{(n)}]$ $(n = 1, 2, \cdots N)$ be applied at the points $A_n$ of $S^*$, the points of application together with $P_0$ lying within a sphere of radius $\epsilon_0$. Let $\mathbf{r}^{(n)} = [x_1^{(n)}, x_2^{(n)}, x_3^{(n)}] = \mathbf{r}(\alpha_n, \beta_n)$ be the position vector of $A_n$. Now consider an $S$ consisting of $N$ non-intersecting closed subregions $S^{(n)}$ of $S^*$ such that each $S^{(n)}$ is simply connected and contains $A_n$ in its interior. Proceeding to the limit in (13) as $S^{(n)}$ is contracted to $A_n$ while $\int_{S^{(n)}} \mathbf{T}\, d\sigma \to \mathbf{T}_n$, we obtain,[28]

$$c\Delta^Q[S] = \mathbf{g}^0 \cdot \sum_{n=1}^{N} \mathbf{T}_n + \mathbf{g}_\alpha^0 \cdot \sum_{n=1}^{N} \mathbf{T}_n\alpha_n + \mathbf{g}_\beta^0 \cdot \sum_{n=1}^{N} \mathbf{T}_n\beta_n + \cdots . \tag{29}$$

The conclusions previously reached, therefore, remain valid for concentrated forces provided the integrals in (18) to (25) are replaced with the corresponding finite sums, and provided $O(\epsilon^2)$, $O(\epsilon^3)$, $O(\epsilon^4)$ are replaced with $O(1)$, $O(\epsilon)$, $O(\epsilon^2)$, respectively.

At this place we discuss an example which may serve to clarify the implications of the theorem established earlier. Consider a bar of the general shape indicated in Figure 2. Let the bar be acted on by the two equal, opposite, and collinear concentrated loads, each of magnitude $L$, the region $S$ consisting of the two points of application $A_1$ and $A_2$. According to conclusion[29] (d), in this instance $\tau_{ij}^Q(\epsilon) = O(\epsilon)$ or smaller, where $Q$ is, say, the fixed interior point shown in Figure 2. Since we may choose the two ends of the bar as close together as we wish, a careless interpretation of (d) may lead to the absurd prediction that the stresses at $Q$, for arbitrarily large fixed magnitudes of the loads, can be kept as small as we wish. The paradox is resolved by observing that the shape of the bar is given once and for all, and that a definite gap, however small, exists between the two ends of the bar. The statement $\tau_{ij}^Q(\epsilon) = O(\epsilon)$, in view of the definition (17), merely implies $|\tau_{ij}^Q(\epsilon)| < M\epsilon$ as $\epsilon \to 0$, where $M$ is a positive number independent of $\epsilon$. Thus, by contracting the load region sufficiently, say toward $A_1$, while maintaining the loading within the $\epsilon$-sphere in equilibrium, $|\tau_{ij}^Q(\epsilon)|$ can be made arbitrarily small. In this process of contraction, however, the end of the bar which carries the point $A_2$, will eventually cease to lie within the contracting $\epsilon$-sphere so that the entire character

[25] $\mathbf{g}_{11}$, for example, at $Q$ has the singularity appropriate to a force-doublet the axis of which is parallel to the $x_1$-axis.

[26] See [3], p. 245.

[27] These examples refer to concentrated forces (see the next section of the present paper).

[28] Note that $S$ here consists of the points $A_n (n = 1, 2, \cdots N)$.

[29] Modified for the case of concentrated forces.

of the loading changes, that end being now free from loading. This example provides additional evidence for the vagueness of the traditional statements of Saint-Venant's principle, quoted in the Introduction.

Hoff [17] pointed out important limitations inherent in certain conventional engineering approximations which are usually based on an appeal to Saint-Venant's principle. These observations are consistent with the results established here. For Saint-Venant's

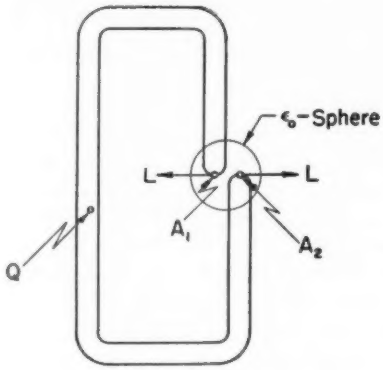


FIG. 2.

principle is a statement about relative orders of magnitude and does not tell us anything about the extent of the region within which a self-equilibrated system of tractions, applied to a portion of the surface of an elastic body, "materially" influences the stress distribution in the body.

In conclusion, it may be well to repeat v. Mises' remark [4] to the effect that the present theorem does not preclude the validity of a stronger Saint-Venant principle for special classes of bodies, such as "thin" plates or shells and "long" cylinders.

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# ON THE EXPANSION OF FUNCTIONS IN TERMS OF THEIR MOMENTS*

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**Summary.** A general method is devised for the reconstruction of functions of a continuous variable from their moments. An analogue is given for functions of a discrete variable. An application is given to the solution of partial differential equations with a given initial condition.

**1. Introduction.** Recently, in some work connected with the lateral spread of showers of cosmic-ray particles in their passage through the atmosphere, the authors were faced with the problem of finding the radial distribution function of the particles with respect to the shower axis. This function was known to be the solution of a partial differential equation, from which, however, only the moments of the function could be conveniently determined. It was then necessary to devise a method for reconstructing the function from its moments. From the physical nature of the problem considered by the authors, it was known that such a function existed; however, situations could arise in which even this knowledge was not available. Criteria for the existence and uniqueness of distribution functions corresponding to a given set of moments having been the subject of extensive study [1], and analytical procedures for the determination of the distribution functions have been described. However, it has also been pointed out [2] that these methods have little value in practice. The only method known to the authors which is well adapted to application has been given by Spencer and Fano [3], but this was not of sufficient generality for our purpose. We therefore developed a method of considerably wider applicability.

The expansion of a function in terms of its moments is a problem which arises not only in methematical physics, but also in many branches of statistics. In the hope that the method may prove useful to workers in fields other than our own, we now give a brief account of the theory in its most general form.

**2. Expansions in terms of the $\delta$-function.** The use of the $\delta$-function has been well established in quantum mechanics [4] and pulse theory; however, the rigour of the mathematical procedures in which it is used has sometimes been questioned [5], and we therefore state at the outset the unambiguous meaning of an equation of the type

$$f^*(x) = w^*(x) \sum_{k=0}^{\infty} a_k(x)\, \delta^{(k)}(x), \tag{1}$$

where the superfix represents the number of differentiations of the $\delta$-function with respect to the arbitrary variable $x$. The first $k$ derivatives of the function $a_k(x)$ must exist for $x = 0$, but $w(x)$ may have any kind of singularity there. Then (1) will be held equivalent to the assertion

$$\int_{-a}^{b} \frac{dx\, q(x) f^*(x)}{w^*(x)} = \sum (-1)^k q_k^{(k)}(0),$$

$$q_k(x) = q(x) a_k(x) \tag{2}$$

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where $q(x)$ is an arbitrary function regular at $x = 0$, and $a$ and $b$ are any positive numbers less than the radius of convergence $\rho$ of the power series

$$q(x) = \sum_{k=0}^{\infty} q^{(k)}(0)x^k/k!. \tag{3}$$

Let

$$f(x) = f^*(x)/w^*(x); \tag{4}$$

then since

$$\int_{-a}^{b} q(x)f(x)\,dx = \sum_{k=0}^{\infty} q^{(k)}(0) \int_{-a}^{b} x^k f(x)\,dx/k!, \tag{5}$$

one has, by comparison with (1) and (2),

$$f(x) = \sum_{k=0}^{\infty} (-1)^k f_{(k)}\, \delta^{(k)}(x)/k!, \tag{6}$$

$$f_{(k)} = \int_{-a}^{b} x^k f(x)\,dx. \tag{7}$$

Thus, any function, integrable in the range $-a < x < b$ can be expanded in this range as a series of derivatives of the $\delta$-function, with coefficients which are proportional to the moments (7) of the function $f(x)$. This result is easily extended to functions of any number of variables.

**3. Expansions in orthogonal polynomials.** The result (6) requires some elaboration in order to provide a practical method for the determination of $f(x)$ when the coefficients $f_{(k)}$ of the series are known. The method which we shall adopt is to make a formal expansion of $\delta^{(k)}(x)$ in series of orthogonal polynomials;* but first we obtain the expansion of the function $f(x)$ which anticipates the result. Let

$$f(x) = w(x) \sum_{n=0}^{\infty} f_n S_n(x); \tag{8}$$

where $S_n(x)$ is a set of polynomials which will be defined presently; and if $f(x)$ has singularities at $x = x_1, x_2, \cdots, x_t$, where $| x_t | < a$, $| x_t | < b$, let $w(x)$ be a "weight-function" with a singularities of the same type, so that $f(x)/w(x)$ is regular for $| x | < a$, $| x | < b$. The polynomials $S_n(x)$ are defined by

$$S_n(x) = \begin{vmatrix} w_{(0)} & w_{(1)} & \cdots & w_{(n)} \\ w_{(1)} & w_{(2)} & \cdots & w_{(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ w_{(n-1)} & w_{(n)} & \cdots & w_{(2n-1)} \\ 1 & x & \cdots & x^n \end{vmatrix} \quad \text{for} \quad n \geq 1, \tag{9}$$

*For the general theory of such polynomials, c.f. Szegö [6]. The cases of Legendre and Laguerre polynomials are known [1].

where $S_0(x) = 1$, and

$$w_{(k)} = \int_{-a}^{b} w(x)x^k \, dx. \tag{10}$$

Then these polynlmials satisfy the orthogonality relations

$$\int_{-a}^{b} w(x)S_m(x)S_n(x) \, dx = N_n\delta_{mn} , \tag{11}$$

where the normality constants are given by

$$N_n = \Delta_n\Delta_{n-1} ,$$

$$\Delta_n = \begin{vmatrix} w_{(0)} & w_{(1)} & \cdots & w_{(n)} \\ w_{(1)} & w_{(2)} & \cdots & w_{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ w_{(n)} & w_{(n+1)} & \cdots & w_{(2n)} \end{vmatrix} \quad \text{for} \quad n \geq 0, \tag{12}$$

$$\Delta_{-1} = 1.$$

There is obviously an expansion of $x^k$ of the form

$$x^k = \sum_{l=0}^{k} \xi_{k,l} S_l(x); \tag{13}$$

hence, for values of $x$ less than the radius of convergence $\rho$ of the power series

$$f(x)/w(x) = \sum_{k=0}^{\infty} \varphi_k x^k \tag{14}$$

which exceeds both $a$ and $b$, there will exist a convergent expansion of the type (8), with coefficients given by

$$f_n = \sum_{k=n}^{\infty} \varphi_k \xi_{k,n} \tag{15}$$

The coefficients $f_n$ are most readily obtained by multiplying (8) by $S_m(x)$, and integrating from $-a$ to $b$; thus:

$$N_n f_n = \int_{-a}^{b} f(x)S_n(x) \, dx$$

$$= \begin{vmatrix} w_{(0)} & w_{(1)} & \cdots & w_{(n)} \\ w_{(1)} & w_{(2)} & \cdots & w_{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ w_{(n-1)} & w_{(n)} & \cdots & w_{(2n-1)} \\ f_{(0)} & f_{(1)} & \cdots & f_{(n)} \end{vmatrix} \quad \text{for} \quad n \geq 1, \tag{16}$$

$$N_0 f_0 = f_{(0)} .$$

The function $f(x)$ has then been expanded in a series of orthogonal polynomials, the coefficients of which are linear combinations of its moments $f_{(k)}$. It is necessary that the quantities $\Delta_n$ defined by (12) should be positive, if a monotonic function $f(x)$ exists. The conditions for this have been discussed in detail by Shohat and Tomarkin [1], Chap. 1.

The result (8) may also be deduced from (1) by expanding $\delta^{(k)}(x)$ in terms of the orthogonal polynomials $S_n(x)$, thus:

$$\delta^{(k)}(x) = w(x)(-1)^k k! \sum_{l=k}^{\infty} \Delta_{l(k)} \, S_l(x)/N_l \tag{17}$$

where $\Delta_{l(k)}$ is the minor of $w_{(l+k)}$ in the last row or column of the determinant $\Delta_l$. Substituting (17) into (1), and interchanging the order of the summations, one then obtains (8), which is therefore one way of interpreting the formula (1).

This procedure is easily adapted to distribution functions $F(n)$ of a discrete variable $n$. Here we consider expansions of the type

$$F(n) = W(n) \sum_{r=0}^{\infty} F_r S_r(n) \tag{18}$$

where

$$S_r(n) = \begin{vmatrix} W_{(0)} & W_{(1)} & \cdots & W_{(r)} \\ \cdots & \cdots & \cdots & \cdots \\ W_{(r-1)} & W_{(r)} & \cdots & W_{(2r-1)} \\ 1 & n & \cdots & n^r \end{vmatrix} \quad \text{for} \quad r \geq 1, \tag{19}$$

$$S_0(n) = 1,$$

and

$$W_{(r)} = \sum_{n=0}^{\infty} W(n) n^r. \tag{20}$$

It is necessary for the existence of the $W_{(r)}$ that $W(n)$ should decrease at least exponentially for large values of $n$.

The determinants $S_r(n)$ satisfy the orthogonality relations

$$\sum_{n=0}^{\infty} W(n) S_q(n) S_r(n) = N_q \delta_{q,r} , \tag{21}$$

where the $N_q$ are defined by equations precisely analogous to (12);

The coefficients $F_r$ in (18) are determined by the relation

$$\sum_{n=0}^{\infty} S_q(n) F(n) = N_q F_q \tag{22}$$

assuming the convergence of the series. Substituting for $S_q(n)$ from (19), this gives

$$N_q F_q = \begin{vmatrix} W_{(0)} & \cdots & W_{(q)} \\ \vdots & & \vdots \\ W_{(q-1)} & \cdots & W_{(2q-1)} \\ F_{(0)} & \cdots & F_{(q)} \end{vmatrix} \quad \text{for} \quad q \geq 1, \tag{23}$$

$$N_0 F_0 = F_{(0)} ,$$

where

$$F_{(q)} = \sum_{n=0}^{\infty} F(n) n^q \tag{24}$$

are the moments of the distribution, assumed to be finite.

In order to ensure that the series (18) should converge, it is necessary to impose rather stringent conditions on $W(n)$, depending on the nature of the function $F(n)$. In practice, if the weight-function $W(n)$ is chosen to approximate fairly closely to $F(n)$, the series is either convergent, or else is asymptotic and equally suitable for computational purposes.

If one has no previous knowledge of the nature of the function $f(x)$ or $F(n)$, but is given only the moments $f_{(k)}$ or $F_{(r)}$ respectively, defined for the range $0 = a \leq x$ (or $n$) $\leq b \leq \infty$, the weight-function should be chosen in the following manner. The asymptotic behaviour of $f_{(k)}$ for large $k$ is compared with that of the moments of a weight-function of the form

$$w(x) = \exp(-Bx^A) \qquad (B > 0,\ A > 0), \tag{25}$$

namely

$$w_{(k)} = (AB^{(k+1)/A})^{-1}\Gamma\{(k+1)/A\} \sim (k/BeA)^{k/A} \quad \text{for} \quad b = \infty, \quad a = 0 \tag{26}$$

$$w_{(k)} \sim b^{k+1} e^{-Bb^A}/(k+1), \qquad b \text{ finite}, \qquad a = 0$$

A numerical comparison of these expressions with the known values enables one to choose a proper value of $A$, and thus to determine the nature of the singularity of $f(x)$, if any, at the origin. Any other singularity will be revealed by a failure of the series (8) to converge for large values of $x$; and in difficult cases it may be necessary to change the origin. A similar procedure may be followed to identify the "singularities" for a discrete variable.

The value of $B$ in (25) should be determined as follows. Assuming that numerical values are available for the first $m + 1$ moments, the series (8) is terminated at the $m$th term, and $B$ chosen so as to give correctly the value of the $(m + 1)$th moment, thus:

$$f_{(m+1)} = \sum_{n=0}^{m} f_n \begin{vmatrix} w_{(0)} & \cdots & w_{(n)} \\ \cdots & \cdots & \cdots \\ w_{(n-1)} & \cdots & w_{(2n-1)} \\ w_{(m+1)} & \cdots & w_{(m+n+1)} \end{vmatrix}. \tag{27}$$

This procedure for the determination of $B$ requires only the solution of an algebraic equation at the $(m + 1)$th degree. If the weight-function is well chosen, it is our experience that a good approximation to $f(x)$ will result from the use of the first few moments only.

**4. Solution of differential equations.** We shall demonstrate the utility of the method described above in the solution of partial differential equations with a special type of boundary condition.

Consider the general equation

$$O_t f(t, x) = O_x f(t, x), \tag{28}$$

where $O_t$ and $O_x$ are operators of the form

$$O_t = \frac{1}{p(t, x)} \frac{\partial}{\partial t} q(t, x)$$

$$O_x = \sum_{k=0}^{\infty} a_k(x) \frac{\partial^k}{\partial x^k}, \tag{29}$$

with the initial condition

$$f(0, x) = \delta(x) \tag{30}$$

By an iteration procedure a solution can be found of the form

$$f(t, x) = \sum_{k=0}^{\infty} \Psi_k(t, x) \tag{31}$$

where

$$\frac{\partial}{\partial t} \{q(t, x)\Psi_{n+1}(t,x)\} = p(t, x) \sum_{k=0}^{\infty} a_k(x) \frac{\partial^k}{\partial x^k} \Psi_n(t, x). \tag{32}$$

One may choose $\psi_0(t, x) = 0$; then $q(t, x)\, \psi_1(t, x)$ must be independent of $t$ and, according to (30),

$$\Psi_1(t, x) = q(0, x)\ \delta(x)/q(t, x) \tag{33}$$

Furthermore,

$$\Psi_2(t, x) = \{q(t, x)\}^{-1} \int_0^t p(t, x) \sum_{k=0}^{\infty} a_k(x) \frac{\partial^k}{\partial x^k} \{q(0, x)\ \delta(x)/q(t, x)\}\, dt, \tag{34}$$

etc.

The solution of (28) obtained in this manner is of the form (1), and can be reduced to the form (6), by use of the general formula

$$a_k(x)\ \delta^{(k)}(x) = \sum_{l=0}^{k} \frac{k!(-1)^l}{l!(k-l)!} a_k^{(l)}(0)\ \delta^{(k-l)}(x). \tag{35}$$

Thus, the moments of the function $f(t, x)$ with respect to the variable $x$ are obtained immediately as functions of $t$; and, using the procedure described in the previous section for the reconstruction of a function from its moments, an explicit solution is obtained of the differential equation which satisfied the boundary condition (30).

The solution $f'(t, x)_x f_p 28)$ for the initial condition

$$f'(0, x) = \lambda(x) \tag{36}$$

is then easily obtained in the form

$$f'(t, x) = \int f(t, x - x')\lambda(x')\, dx'. \tag{37}$$

An almost identical method of solution may be devised for equations of the type

$$O_t f(t, x_1, \cdots, x_m) = O_{x_1, \cdots, x_m} f(t, x_1, \cdots, x_m)$$

which, in the application to cosmic ray shower theory, we have actually [7] succeeded in solving.

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# LINE LOAD APPLIED ALONG GENERATORS OF THIN-WALLED CIRCULAR CYLINDRICAL SHELLS OF FINITE LENGTH*

BY

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**Summary.** Donnell's differential equations of the thin circular cylindrical shell are integrated in the case when the loads are radial forces or circumferential moments distributed sinusoidally along a generator. Closed form expressions are obtained for the displacements, internal moments, and the membrane stresses. In addition, loads distributed uniformly along a segment of a generator and concentrated loads are discussed and radial forces are combined into a longitudinal moment.

**1. Introduction.** One of the most common elements of structures and machinery is the thin-walled circular cylindrical shell. When other elements are attached to it, forces and moments are likely to be transmitted to the shell across the areas of contact because of gravity and inertia effects, and in consequence of thermal expansion. These forces and moments can often be represented, with an accuracy sufficient for engineering purposes, as loads distributed along a short segment of a generator. For this reason it is of practical interest to investigate the deformations and the stresses of thin-walled circular cylindrical shells loaded along generators.

The problem of loads along generators was first solved by Finsterwalder [1] in 1932 in his investigation of the disturbance stresses arising at the free edge of a circular cylindrical shell-type roof structure. In his approximate theory the longitudinal moment $M_x$ and the torque $M_{x\varphi}$ in the shell were disregarded and a single partial differential equation in the circumferential moment $M_\varphi$ was derived. This approach was further simplified by Schorer [2] in 1935 whose differential equation in $M_\varphi$ consisted of only two terms. In the same year Dischinger [3] derived a rigorous solution. In order to reduce the large amount of work necessary to obtain numerical results from the rigorous theory, Aas Jakobsen [4] developed an iteration procedure in 1939 through which the equilibrium conditions could be expressed with sufficient accuracy in terms of the radial displacement $w$ alone. In 1941 he showed [5] how this approach could be used in the calculation of the effects of concentrated loads.

The concentrated load problem was attacked independently by S. W. Yuan [6] in 1946 who solved Donnell's single eighth order differential equation in $w$ for an indefinitely long shell. The surface loading was represented by a Fourier series in the circumferential direction and by a Fourier integral in the longitudinal direction. The method of images was then used to obtain a solution for the cylinder of finite length. In the same year Odqvist [7] gave a closed form solution for the deflections of finite cylinders subjected to sinusoidally distributed line loads and infinite series solutions for the deflections of

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finite cylinders subjected to concentrated loads. These results were obtained from Schorer's equation.

Although the work of German and Scandinavian civil engineers seems to indicate that the moment $M_x$ and the torque $M_{x\varphi}$ are unimportant in the balance of forces and moments in a reasonably long shell-type roof subjected to distributed loads, the same conclusion need not necessarily hold when line loads or concentrated loads are applied to the structure. Indeed, in the experiments performed by Schoessow and Kooistra [8], in which forces and moments were transmitted to a shell by means of comparatively large-diameter pipes, the ratio of the maximum longitudinal bending stress to the maximum circumferential bending stress was found to be close to one-half. Therefore some doubt arises regarding the range of applicability of Schorer's equation. But Odqvist's paper, based on this equation, is the only one to present results from which the stresses in the shell can be obtained without extensive calculations. This situation prompted the development of the new solution given in this paper.

In the derivations that follow the stipulation of the conditions of equilibrium in the forms presented by Love [9], Flügge [10], Biezeno and Grammel [11], and Timoshenko [12] is replaced by the simplified statement proposed by Donnell [13] and recently recommended by Batdorf [14]. In addition to the eighth-order differential equation in the single dependent variable $w$, used and re-derived by Yuan, two fourth order equations are satisfied rigorously. The dependent variables are $u$ and $w$ in one equation, and $v$ and $w$ in the other; $w$ is the radial displacement, $u$ the longitudinal displacement, and $v$ the circumferential displacement. Closed form solutions are given for the displacement, moment, and membrane stress quantities arising from sinusoidally distributed line loads. Series of these solutions are capable of representing the effects of loads constant over segments of generators as well as those of concentrated loads.

**2. Basic equations.** In the absence of surface and body forces Donnell's [13] equilibrium conditions of an element of a thin-walled circular cylindrical shell can be stated in the following form:

$$\nabla^8 w + 4K^4 w_{,xxxx} = 0, \tag{1}$$

$$\nabla^4 u = \nu w_{,xxx} - w_{,x\varphi\varphi}\,, \tag{2}$$

$$\nabla^4 v = (2 + \nu) w_{,xx\varphi} + w_{,\varphi\varphi\varphi}\,, \tag{3}$$

where the subscripts following a comma indicate differentiation,

$$4K^4 = 12(1 - \nu^2)(a/h)^2, \qquad D = Eh^3/[12(1 - \nu^2)], \tag{4}$$

and $\nabla^2$ is Laplace's operator

$$\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial \varphi^2). \tag{5}$$

The non-dimensional distances and displacements are defined by the equations

$$x = (x^*/a), \qquad u = (u^*/a), \qquad v = (v^*/a), \qquad w = (w^*/a), \tag{6}$$

and $x^*$ is the distance measured in the axial direction along a generator, $a\varphi$ that measured around the circumference, and $u^*$, $v^*$, and $w^*$ are the displacements in the axial, circumferential, and radial directions, respectively, as shown in Fig. 1a. The remaining geometrical and physical quantities are $a$, the mean radius of the shell; $h$, the thickness of

the shell; $L$, the length of the shell; $E$, Young's modulus of elasticity; and $\nu$, Poisson's ratio.

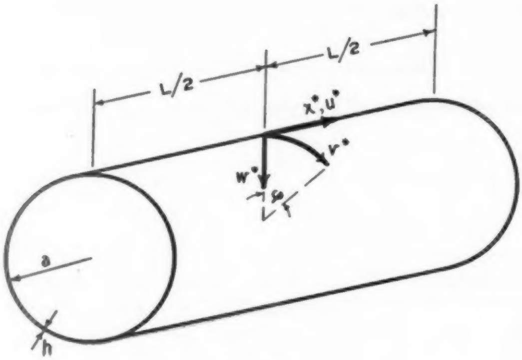


Fig. 1a.

In Donnell's approximation and with the present sign convention the strains $\epsilon$ and the curvatures $\kappa$ of the median surface of the shell are defined as

$$\begin{aligned} \epsilon_x &= u_{,x}\,, & \kappa_x &= (1/a)w_{,xx}\,, \\ \epsilon_\varphi &= v_{,\varphi} - w, & \kappa_\varphi &= (1/a)w_{,\varphi\varphi}\,, \\ \gamma_{x\varphi} &= u_{,\varphi} + v_{,x}\,, & \kappa_{x\varphi} &= (1/a)w_{,x\varphi}\,. \end{aligned} \tag{7}$$

The membrane stresses can be given as

$$\begin{aligned} \sigma_x &= [E/(1-\nu^2)](\epsilon_x + \nu\epsilon_\varphi) = [E/(1-\nu^2)](u_{,x} + \nu v_{,\varphi} - \nu w), \\ \sigma_\varphi &= [E/(1-\nu^2)](\nu\epsilon_x + \epsilon_\varphi) = [E/(1-\nu^2)](\nu u_{,x} + v_{,\varphi} - w), \\ \tau_{x\varphi} &= [E/2(1+\nu)]\gamma_{x\varphi} = [E/2(1+\nu)](u_{,\varphi} + v_{,x}). \end{aligned} \tag{8}$$

The moment-resultants per unit length are indicated as right-hand vectors in Fig. 1b. They are

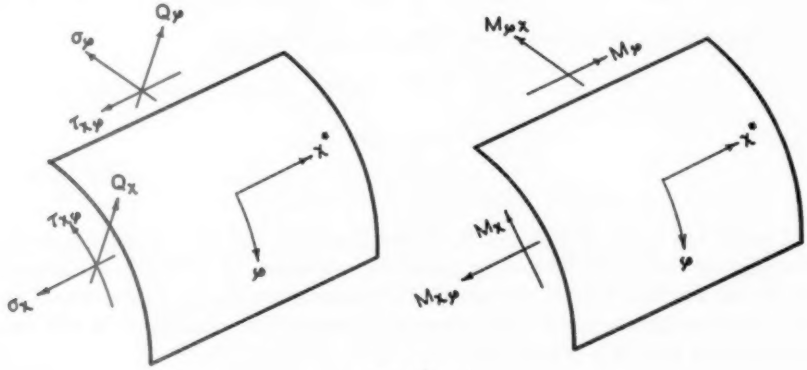


Fig. 1b.

$$M_x = -D(\kappa_x + \nu\kappa_\varphi) = -(D/a)(w_{,xx} + \nu w_{,\varphi\varphi}),$$

$$M_\varphi = -D(\nu\kappa_x + \kappa_\varphi) = -(D/a)(w_{,\varphi\varphi} + \nu w_{,xx}), \tag{9}$$

$$M_{x\varphi} = -M_{\varphi x} = (1 - \nu)D\kappa_{x\varphi} = [(1 - \nu)D/a]w_{,x\varphi} .$$

The shear forces per unit length which accompany these moments and act in the radial direction are

$$Q_x = (1/a)(M_{x,x} + M_{\varphi x,\varphi}) = -(D/a^2)(w_{,xxx} + w_{,x\varphi\varphi})$$
$$Q_\varphi = (1/a)(M_{\varphi,\varphi} - M_{x\varphi,x}) = -(D/a^2)(w_{,\varphi\varphi\varphi} + w_{,xx\varphi}) \tag{10}$$

If the equivalent shear $(1/a)M_{\varphi x,x}$ corresponding to the distributed torque is added to $Q_\varphi$ in agreement with Kirchhoff's suggestion (see, for instance, p. 90 of Reference 12), the total effective shear force per unit length is

$$Q_{\varphi.\text{eff.}} = -(D/a^2)[w_{,\varphi\varphi\varphi} + (2 - \nu)w_{,xx\varphi}] \tag{11}$$

**3. Sinusoidally distributed radial force symmetric with respect to $x = 0$.** It is assumed that the edges $x^* = (L/2)$ are simply supported. This condition implies that the radial displacement $w$ vanishes and so do the axial membrane stress and the axial moment resultant, but not the membrane shear stress. If the expressions just given for these quantities are considered, the simple-support condition can be reduced to the following requirements:

$$u_{,x} = v = w = w_{,xx} = 0, \quad \text{when} \quad x = \pm(1/2)(L/a). \tag{12}$$

The load is applied along the generator $\varphi = 0$ in the radial inward direction and is $2V \cos nx$ with $n = m\pi(a/L)$ and $m$ an odd integer. With $-\pi \leq \varphi \leq \pi$, the solution is assumed as

$$w_{\text{tot}} = w(\varphi) + w(2\pi - \varphi) + w(2\pi + \varphi) + w(4\pi - \varphi) + w(4\pi + \varphi) + \cdots , \tag{13a}$$

$$u_{\text{tot}} = u(\varphi) + u(2\pi - \varphi) + u(2\pi + \varphi) + u(4\pi - \varphi) + u(4\pi + \varphi) + \cdots , \tag{13b}$$

$$v_{\text{tot}} = v(\varphi) - v(2\pi - \varphi) + v(2\pi + \varphi) - v(4\pi - \varphi) + v(4\pi + \varphi) - \cdots , \tag{13c}$$

where the dependence on $x$ is not shown but is understood, and

$$w(\varphi) = Ae^{p\varphi} \cos nx, \tag{14a}$$

$$u(\varphi) = Be^{p\varphi} \sin nx, \tag{14b}$$

$$v(\varphi) = Ce^{p\varphi} \cos nx, \tag{14c}$$

where $A$ and $n$ are assumed real and $B$, $C$, and $p$ may be complex. These displacements can be imagined to be contributed by two fictitious sheets of width $L$, each beginning at $\varphi = 0$ and winding around the actual shell indefinitely many times, one clockwise and the other counterclockwise. Only those solutions will be considered in this analysis in which the real part of $p$ is negative.

It can be seen that the assumed deflections satisfy the boundary conditions at

$x = \pm(1/2)(L/a)$. In the circumferential direction they are continuous everywhere except at $\varphi = 0$. By a suitable choice of the conditions along the generator $\varphi = 0$, continuity can be enforced there and the prescribed line load can be applied to the shell. Worked-out numerical examples have shown that in most cases the first term of the infinite series suffices for a satisfactory approximation to the deflections, and two terms are needed only infrequently.

The boundary conditions at $\varphi = 0$ are:

(1) For reasons of symmetry

$$w_{,\varphi} = 0 \qquad \text{when} \qquad \varphi = 0. \tag{15}$$

(2) To eliminate gaps caused by the circumferential displacements

$$v = 0 \qquad \text{when} \qquad \varphi = 0. \tag{16}$$

(3) The shear stress $\tau_{x\varphi}$ acts in the same direction along the edges of the two sheets at $\varphi = 0$. In the absence of a longitudinal line load, $\tau_{x\varphi}$ must vanish. Hence

$$u_{,\varphi} + v_{,x} = 0 \qquad \text{when} \qquad \varphi = 0.$$

But $v_{,x}$ vanishes at $\varphi = 0$ in consequence of (16). Therefore the requirement reduces to

$$u_{,\varphi} = 0 \qquad \text{when} \qquad \varphi = 0. \tag{17}$$

(4) Finally the total effective shear force $Q_{\varphi_{eff}}$, given in (11), must be equal to one-half the applied radial load. However the second term in the brackets vanishes in consequence of (15) with the result that

$$(D/a^2)w_{,\varphi\varphi\varphi} = V \cos nx. \tag{18}$$

**4. Enforcement of the differential equations.** Substitution into (1) of the expression assumed for $w$ yields

$$[(p^2 - n^2)^4 + 4K^4n^4]w = 0 \tag{19}$$

which must be satisfied identically. Hence

$$p^2 = n^2 + \sqrt{2}\,(-1)^{1/4}nK.$$

If the fourth root of $-1$ is taken as $(\sqrt{2}/2)(1 - i)$ then

$$p_1^2 = n^2 + (1 - i)nK, \tag{20}$$

and if $p_1$ is written as

$$p_1 = -\alpha_1 + i\beta_1 ,$$

where $\alpha_1$ and $\beta_1$ are positive real numbers, one has

$$\alpha_1^2 - \beta_1^2 = n(n + K), \tag{20a}$$

$$\alpha_1\beta_1 = (1/2)nK, \tag{20b}$$

$$\beta_1 = (1/2)(nK/\alpha_1). \tag{20c}$$

Finally

$$\alpha_1^2 = (n/2)\{(n + K) + [(n + K)^2 + K^2]^{1/2}\}. \tag{20d}$$

The square root is taken with the positive sign because $\alpha_1$ must be real by assumption.

When the choice is

$$(-1)^{1/4} = (\sqrt{2}/2)(-1 - i),$$

the value of $p$ is designated as $p_2$. The above equations are replaced by

$$p_2 = -\alpha_2 + i\beta_2 ,$$

and

$$\alpha_2^2 - \beta_2^2 = n(n - K), \tag{21a}$$

$$\alpha_2\beta_2 = (1/2)nK, \tag{21b}$$

$$\beta_2 = (1/2)(nK/\alpha_2), \tag{21c}$$

$$\alpha_2^2 = (n/2)\{(n - K) + [(n - K)^2 + K^2]^{1/2}\}. \tag{21d}$$

The conjugate solutions, corresponding to the remaining two fourth roots of $-1$, namely

$$p_3 = -\alpha_1 - i\beta_1 , \qquad p_4 = -\alpha_2 - i\beta_2$$

need not be discussed here in detail because they are governed by the same relationships as $p_1$ and $p_2$, respectively.

Of course the solutions for $\alpha_1$ and $\alpha_2$ of (20d) and (21d) can be negative as well as positive. The negative values are rejected because they lead to positive real parts of the exponent $p$ and thus yield displacements which increase without bounds as $\varphi$ increases.

If the deflections assumed in (14) for $u$ and $w$ are substituted in equilibrium condition (2) one obtains

$$B(p^2 - n^2)^2 = An(\nu n^2 + p^2). \tag{22}$$

Consequently

$$B = A\nu n^3[1/(p^2 - n^2)^2] + An[p^2/(p^2 - n^2)^2]. \tag{23}$$

When $p = p_1$ one has

$$(p_1^2 - n^2)^2 = -i(2K^2n^2), \qquad 1/(p_1^2 - n^2)^2 = +[i/2K^2n^2],$$

$$p_1^2/(p_1^2 - n^2)^2 = (1/2Kn) + i[(n + K)/2K^2n].$$

Equation (23) becomes

$$B = (A/2K^2)\{K + i[(1 + \nu)n + K]\} \qquad \text{when} \qquad p = p_1 . \tag{24}$$

When $p = p_2$ the equations following (23) become

$$(p_2^2 - n^2)^2 = i(2K^2n^2), \qquad 1/(p_2^2 - n^2)^2 = -[i/2K^2n^2],$$

$$p_2^2/(p_2^2 - n^2)^2 = -(1/2Kn) - i[(n - K)/2K^2n]$$

and thus

$$B = (A/2K^2)\{-K - i[(1 + \nu)n - K]\} \qquad \text{when} \qquad p = p_2 . \tag{25}$$

Substitutions from (14a) and (14*c*) into (3) result in

$$C(p^2 - n^2)^2 = Ap[p^2 - (2 + \nu)n^2]. \tag{26}$$

Manipulations similar to the ones shown lead to

$$C = (A/2K^2n)\{-\alpha_1 K + \beta_1[(1 + \nu)n - K] + i\{\beta_1 K + \alpha_1[(1 + \nu)n - K]\}\} \tag{27}$$

when $p = p_1$ and to

$$C = (A/2K^2n)\{\alpha_2 K - \beta_2[(1 + \nu)n + K] + i\{-\beta_2 K - \alpha_2[(1 + \nu)n + K]\}\} \tag{28}$$

when $p = p_2$ .

Instead of making use of all the four exponents $p_1$ to $p_4$ , it is more convenient to give the radial deflections $w$ in the following form:

$$w = [(A_1 R + A_2 I)e^{p_1\varphi} + (A_3 R + A_4 I)e^{p_2\varphi}] \cos nx. \tag{29}$$

Here the expressions in parentheses are operators with $R$ signifying the real part and $I$ the imaginary part of the complex function upon which the operation is performed. Equation (29) can be written explicitly as

$$w = [A_1 e^{-\alpha_1\varphi} \cos \beta_1\varphi + A_2 e^{-\alpha_1\varphi} \sin \beta_1\varphi + A_3 e^{-\alpha_2\varphi} \cos \beta_2\varphi + A_4 e^{-\alpha_2\varphi} \sin \beta_2\varphi] \cos (nx). \tag{30}$$

From (14b), (24), and (25) one obtains

$$\begin{aligned} u = {} & (A_1 R + A_2 I)(1/2K^2)\{K + i[(1 + \nu)n + K]\}e^{-\alpha_1\varphi}(\cos \beta_1\varphi + i \sin \beta_1\varphi) \sin (nx) \\ & + (A_3 R + A_4 I)(1/2K^2)\{-K - i[(1 + \nu)n - K]\}e^{-\alpha_2\varphi}(\cos \beta_2\varphi + i \sin \beta_2\varphi) \sin (nx). \end{aligned} \tag{31}$$

When the operations indicated are performed, (31) becomes

$$\begin{aligned} u = {} & (A_1/2K^2)e^{-\alpha_1\varphi}\{K \cos \beta_1\varphi - [(1 + \nu)n + K] \sin \beta_1\varphi\} \sin nx \\ & + (A_2/2K^2)e^{-\alpha_1\varphi}\{[(1 + \nu)n + K] \cos \beta_1\varphi + K \sin \beta_1\varphi\} \sin nx \\ & + (A_3/2K^2)e^{-\alpha_2\varphi}\{-K \cos \beta_2\varphi + [(1 + \nu)n - K] \sin \beta_2\varphi\} \sin nx \\ & + (A_4/2K^2)e^{-\alpha_2\varphi}\{-[(1 + \nu)n - K] \cos \beta_2\varphi - K \sin \beta_2\varphi\} \sin nx. \end{aligned} \tag{32}$$

The circumferential displacement is found in a similar manner:

$$\begin{aligned} v = {} & (A_1/2K^2n)e^{-\alpha_1\varphi}\{\{-\alpha_1 K + \beta_1[(1 + \nu)n - K]\} \cos \beta_1\varphi \\ & - \{\beta_1 K + \alpha_1[(1 + \nu)n - K]\} \sin \beta_1\varphi\}(\cos nx) \\ & + (A_2/2K^2n)e^{-\alpha_1\varphi}\{\{\beta_1 K + \alpha_1[(1 + \nu)n - K]\} \cos \beta_1\varphi \\ & + \{-\alpha_1 K + \beta_1[(1 + \nu)n - K]\} \sin \beta_1\varphi\}(\cos nx) \\ & + (A_3/2K^2n)e^{-\alpha_2\varphi}\{\{\alpha_2 K - \beta_2[(1 + \nu)n + K]\} \cos \beta_2\varphi \\ & + \{\beta_2 K + \alpha_2[(1 + \nu)n + K]\} \sin \beta_2\varphi\}(\cos nx) \\ & + (A_4/2K^2n)e^{-\alpha_2\varphi}\{\{-\beta_2 K - \alpha_2[(1 + \nu)n + K]\} \cos \beta_2\varphi \\ & + \{\alpha_2 K - \beta_2[(1 + \nu)n + K]\} \sin \beta_2\varphi\}(\cos nx). \end{aligned} \tag{33}$$

**5. Enforcement of the boundary conditions at $\varphi = 0$.** From (29)

$$w_{,\varphi} = [(A_1R + A_2I)p_1e^{p_1\varphi} + (A_3R + A_4I)p_2e^{p_2\varphi}] \cos(nx).$$

At $\varphi = 0$ the function $w_{,\varphi}$ must vanish. Hence the requirement contained in (15) leads to

$$(A_1R + A_2I)p_1 + (A_3R + A_4I)p_2 = 0. \tag{34}$$

This can also be written as

$$-\alpha_1A_1 + \beta_1A_2 - \alpha_2A_3 + \beta_2A_4 = 0. \tag{35}$$

From (16) and (33) one obtains

$$\{-\alpha_1K + \beta_1[(1 + \nu)n - K]\}A_1 + \{\beta_1K + \alpha_1[(1 + \nu)n - K]\}A_2$$
$$+ \{\alpha_2K - \beta_2[(1 + \nu)n + K]\}A_3 + \{-\beta_2K - \alpha_2[(1 + \nu)n + K]\}A_4 = 0. \tag{36}$$

To satisfy the conditions expressed by (17), $u_{,\varphi}$ must be calculated:

$$u_{,\varphi} = (1/2K^2)(A_1R + A_2I)\{K + i[(1 + \nu)n + K]\}p_1e^{p_1\varphi} \sin(nx)$$
$$+ (1/2K^2)(A_3R + A_4I)\{-K - i[(1 + \nu)n - K]\}p_2e^{p_2\varphi} \sin(nx).$$

Evaluation of this expression for $\varphi = 0$ results in

$$\{-\alpha_1K - \beta_1[(1 + \nu)n + K]\}A_1 + \{\beta_1K - \alpha_1[(1 + \nu)n + K]\}A_2$$
$$+ \{\alpha_2K + \beta_2[(1 + \nu)n - K]\}A_3 + \{-\beta_2K + \alpha_2[(1 + \nu)n - K]\}A_4 = 0. \tag{37}$$

From these conditions $A_1$, $A_2$, and $A_3$ can be expressed in terms of $A_4$:

$$A_1 = -\left(\frac{\alpha_1 - \beta_1}{\alpha_1^2 + \beta_1^2}\right)\left(\frac{\alpha_2^2 + \beta_2^2}{\alpha_2 - \beta_2}\right)A_4, \tag{38}$$

$$A_2 = \left(\frac{\alpha_1 + \beta_1}{\alpha_1^2 + \beta_1^2}\right)\left(\frac{\alpha_2^2 + \beta_2^2}{\alpha_2 - \beta_2}\right)A_4, \tag{39}$$

$$A_3 = \left(\frac{\alpha_2 + \beta_2}{\alpha_2 - \beta_2}\right)A_4. \tag{40}$$

In the calculation of the shear, $w_{,\varphi\varphi\varphi}$ is needed. From (29) one obtains

$$w_{,\varphi\varphi\varphi} = [(A_1R + A_2I)p_1^3e^{p_1\varphi} + (A_3R + A_4I)p_2^3e^{p_2\varphi}] \cos(nx).$$

At $\varphi = 0$ this equation becomes

$$w_{,\varphi\varphi\varphi} = [(A_1R + A_2I)p_1^3 + (A_3R + A_4I)p_2^3] \cos(nx). \tag{41}$$

Evaluation of the operations indicated in (41) and substitution in (18) result in

$$V = -(D/a^2)\{(\alpha_1^3 - 3\alpha_1\beta_1^2)A_1 + (\beta_1^3 - 3\beta_1\alpha_1^2)A_2$$
$$+ (\alpha_2^3 - 3\alpha_2\beta_2^2)A_3 + (\beta_2^3 - 3\beta_2\alpha_2^2)A_4\}. \tag{42}$$

If the expressions given in (38), (39), and (40) are substituted into (42), manipulations yield

$$V = 4Kn(D/a^2)\left(\frac{\alpha_2^2 + \beta_2^2}{\alpha_2 - \beta_2}\right)A_4 . \tag{43}$$

This can be solved for $A_4$ :

$$A_4 = \frac{V}{D}\frac{a^2}{4Kn}\left(\frac{\alpha_2 - \beta_2}{\alpha_2^2 + \beta_2^2}\right). \tag{44}$$

The other three constants can also be expressed directly in terms of the applied load:

$$A_1 = -\frac{V}{D}\frac{a^2}{4Kn}\left(\frac{\alpha_1 - \beta_1}{\alpha_1^2 + \beta_1^2}\right), \tag{45}$$

$$A_2 = \frac{V}{D}\frac{a^2}{4Kn}\left(\frac{\alpha_1 + \beta_1}{\alpha_1^2 + \beta_1^2}\right), \tag{46}$$

$$A_3 = \frac{V}{D}\frac{a^2}{4Kn}\left(\frac{\alpha_2 + \beta_2}{\alpha_2^2 + \beta_2^2}\right). \tag{47}$$

The expressions for the displacements $u$, $v$, and $w$ are then as follows:

$$\begin{aligned} u = \frac{Va^2}{8K^3nD}\Bigg\{&\frac{e^{-\alpha_1\varphi}}{\alpha_1^2 + \beta_1^2}\{[2\beta_1K + n(1 + \nu)(\alpha_1 + \beta_1)] \cos \beta_1\varphi \\ &+ [2\alpha_1K + n(1 + \nu)(\alpha_1 - \beta_1)] \sin \beta_1\varphi\} \\ &+ \frac{e^{-\alpha_2\varphi}}{\alpha_2^2 + \beta_2^2}\{[-2\beta_2K - n(1 + \nu)(\alpha_2 - \beta_2)] \cos \beta_2\varphi \\ &+ [-2\alpha_2K + n(1 + \nu)(\alpha_2 + \beta_2)] \sin \beta_2\varphi\}\Bigg\} \sin (nx), \end{aligned} \tag{48}$$

$$\begin{aligned} v = \frac{Va^2}{8K^3n^2D}\{&e^{-\alpha_1\varphi}\{n(1 + \nu) \cos \beta_1\varphi + [n(1 + \nu) - 2K] \sin \beta_1\varphi\} \\ &+ e^{-\alpha_2\varphi}\{-n(1 + \nu) \cos \beta_2\varphi + [n(1 + \nu) + 2K] \sin \beta_2\varphi\}\} \cos (nx), \end{aligned} \tag{49}$$

$$\begin{aligned} w = \frac{Va^2}{4KnD}\Bigg\{&\frac{e^{-\alpha_1\varphi}}{\alpha_1^2 + \beta_1^2}\{-(\alpha_1 - \beta_1) \cos \beta_1\varphi + (\alpha_1 + \beta_1) \sin \beta_1\varphi\} \\ &+ \frac{e^{-\alpha_2\varphi}}{\alpha_2^2 + \beta_2^2}\{(\alpha_2 + \beta_2) \cos \beta_2\varphi + (\alpha_2 - \beta_2) \sin \beta_2\varphi\}\Bigg\} \cos (nx). \end{aligned} \tag{50}$$

**6. Constant radial force over segment of generator.** The results obtained can be used in the calculation of the displacements and stresses of cylinders subjected to symmetric radial forces of all kinds. For instance, when the inward radial load is stipulated as

$$(P^*/2\delta) \quad \text{when} \quad |x^*| \le \delta \qquad \text{and as} \qquad 0 \quad \text{when} \quad (L/2) \ge |x^*| > \delta \tag{51}$$

it can be represented by the Fourier series

$$\sum_{m=1,3,\cdots}^{\infty} P^{*(m)} \cos (m\pi ax/L)$$

with the coefficient $P^{*(m)}$ defined as

$$P^{*(m)} = (2P^*/m\pi\delta) \sin (m\pi\delta/L) \qquad m = 1, 3, \cdots . \tag{52}$$

The displacements $u$, $v$, and $w$ can then be calculated from (48), (49), and (50), respectively, if $V$ is replaced by $(1/2)P^{*(m)}$ and the resulting expressions are summed over $m$. Naturally the summation indicated in (13) must also be carried out if the first term alone does not give sufficient accuracy. However, this is seldom the case. The summation causes no difficulties because the right-hand members of (13) are geometric series.

If the coefficient $b_m$ is defined as

$$b_m = (P^*aL/4\pi^2 DK\delta)(1/m^2) \sin (m\pi\delta/L) \tag{53}$$

substitutions and simplifications yield the following expressions for the $m$th components of the moments and the membrane stresses:

$$\begin{aligned} M_x^{(m)}/[(-D/a)nb_m \cos nx] = {} & \frac{e^{-\alpha_1\varphi}}{\alpha_1^2 + \beta_1^2} \{[-2\nu K\alpha_1 + n(1-\nu)(\alpha_1 - \beta_1)] \cos \beta_1\varphi \\ & + [2\nu K\beta_1 - n(1-\nu)(\alpha_1 + \beta_1)] \sin \beta_1\varphi\} \\ & + \frac{e^{-\alpha_2\varphi}}{\alpha_2^2 + \beta_2^2} \{[-2\nu K\alpha_2 - n(1-\nu)(\alpha_2 + \beta_2)] \cos \beta_2\varphi \\ & + [2\nu K\beta_2 - n(1-\nu)(\alpha_2 - \beta_2)] \sin \beta_2\varphi\}, \end{aligned} \tag{54}$$

$$\begin{aligned} M_\varphi^{(m)}/[(-D/a)nb_m \cos nx] = {} & \frac{e^{-\alpha_1\varphi}}{\alpha_1^2 + \beta_1^2} \{[-2K\alpha_1 - n(1-\nu)(\alpha_1 - \beta_1)] \cos \beta_1\varphi \\ & + [2K\beta_1 + n(1-\nu)(\alpha_1 + \beta_1)] \sin \beta_1\varphi\} \\ & + \frac{e^{-\alpha_2\varphi}}{\alpha_2^2 + \beta_2^2} \{[-2K\alpha_2 + n(1-\nu)(\alpha_2 + \beta_2)] \cos \beta_2\varphi \\ & + [2K\beta_2 + n(1-\nu)(\alpha_2 - \beta_2)] \sin \beta_2\varphi\}, \end{aligned} \tag{55}$$

$$\begin{aligned} M_{x\varphi}^{(m)}/[(D/a)(1-\nu)nb_m \sin nx] = {} & e^{-\alpha_1\varphi}[-\cos \beta_1\varphi + \sin \beta_1\varphi] \\ & + e^{-\alpha_2\varphi}[\cos \beta_2\varphi + \sin \beta_2\varphi], \end{aligned} \tag{56}$$

$$\begin{aligned} & \sigma_x^{(m)}/[(E/2K^2)nb_m \cos nx] \\ & = \frac{e^{-\alpha_1\varphi}}{\alpha_1^2 + \beta_1^2} \{[n\alpha_1 + (n + 2K)\beta_1] \cos \beta_1\varphi + [(n + 2K)\alpha_1 - n\beta_1] \sin \beta_1\varphi\} \\ & + \frac{e^{-\alpha_2\varphi}}{\alpha_2^2 + \beta_2^2} \{[-n\alpha_2 + (n - 2K)\beta_2] \cos \beta_2\varphi + [(n - 2K)\alpha_2 + n\beta_2] \sin \beta_2\varphi\}, \end{aligned} \tag{57}$$

$$\sigma_{\varphi}^{(m)}/[(E/2K^2)n^2 b_m \cos nx] = \frac{e^{-\alpha_1\varphi}}{\alpha_2^2 + \beta_2^2}\{-(\alpha_1 + \beta_1)\cos\beta_1\varphi + (-\alpha_1 + \beta_1)\sin\beta_1\varphi\}$$

$$+ \frac{e^{-\alpha_2\varphi}}{\alpha_2^2 + \beta_2^2}\{(\alpha_2 - \beta_2)\cos\beta_2\varphi - (\alpha_2 + \beta_2)\sin\beta_2\varphi\}, \quad (58)$$

$$\tau_{x\varphi}^{(m)}/[(E/2K^2)nb_m \sin nx] = -e^{-\alpha_1\varphi}\{\cos\beta_1\varphi + \sin\beta_1\varphi\}$$

$$+ e^{-\alpha_2\varphi}\{\cos\beta_2\varphi - \sin\beta_2\varphi\}. \quad (59)$$

When the load is concentrated at $x = 0$, the distance $\delta$ approaches zero. In that case

$$P^{(m)} = (2P^*/L), \quad (60)$$

and

$$b_m = (P^*aL/4\pi DK)(1/m). \quad (61)$$

**7. Sinusoidally distributed radial force antisymmetric with respect to $x = 0$.** The load is applied along the generator $\varphi = 0$ in the radial inward direction and is $2V\sin nx$ with $n = m\pi(a/L)$ and $m$ an even integer. The solution is assumed in the form of the series (13) with

$$w = Ae^{p\varphi} \sin nx, \quad (62a)$$

$$u = Be^{p\varphi} \cos nx, \quad (62b)$$

$$v = Ce^{p\varphi} \sin nx, \quad (62c)$$

where again $A$ and $n$ are assumed real while $B$, $C$, and $p$ may be complex. The boundary conditions at $x = \pm(1/2)(L/a)$ and at $\varphi = 0$ remain unchanged except that

$$(D/a^2)w_{,\varphi\varphi\varphi} = V \sin nx. \quad (63)$$

Solution of this problem leads to the same expressions for the constants $A_1$ to $A_4$ as obtained earlier and presented in (44) to (47). Similarly the expressions given in (49) and (50) for the displacements $v$ and $w$ are unchanged except for the replacement of the factors ($\cos nx$) by ($\sin nx$); also in expression (48) for $u$ the factor ($\sin nx$) must be replaced by ($-\cos nx$).

**8. Loading by longitudinal moment.** When the radial inward load along the generator $\varphi = 0$ is stipulated as

$$\begin{array}{lll} M_x^*/\delta^2 & \text{when} & 0 \leq x^* \leq \delta, \\ -M_x^*/\delta^2 & \text{when} & -\delta \leq x^* \leq 0, \\ 0 & \text{when} & (L/2) \geq |x^*| > \delta, \end{array} \quad (64)$$

where $M_x^*$ is the applied longitudinal moment to which the line loading is statically equivalent, the loading can be represented by the Fourier series

$$\sum_{m=2,4,\cdots}^{\infty} M_x^{*(m)} \sin(m\pi ax/L).$$

The coefficient $M_x^{*(m)}$ is defined as

$$M_x^{*(m)} = (4M_x^*/m\pi\delta^2)[1 - \cos(m\pi\delta/L)] \qquad m = 2, 4, \cdots . \tag{65}$$

The deflections $u$, $v$, and $w$ can be calculated from (48), (49), and (50), respectively, if $V$ is replaced by $(1/2)M_x^{*(m)}$, the changes in sign and in the trigonometric expressions stated in the preceding article are carried out, and the resulting expressions are summed over $m$. Naturally the summation indicated in (13) must also be carried out if the first term alone does not give sufficient accuracy. However, this is seldom the case.

If the coefficient $b_m$ is now defined as

$$b_m = (M_x^* aL/2\pi^2 DK\delta^2)(1/m^2)[1 - \cos(m\pi\delta/L)] \qquad m = 2, 4, \cdots \tag{66}$$

the $m$th components of the moments and of the membrane stresses are again given by equations (54) to (59) provided all factors ($\cos nx$) are replaced by ($\sin nx$) and all factors ($\sin nx$) by ($-\cos nx$).

When the moment is concentrated at $x = 0$, the distance $\delta$ approaches zero. In that case

$$M_x^{(m)} = (2\pi M_x^*/L^2)m \tag{67}$$

and

$$b_m = (M_x^*/4DK)(a/L). \tag{68}$$

**9. Sinusoidally distributed circumferential moment symmetric with respect to $x = 0$.** This problem differs from the symmetric sinusoidally distributed force problem only in the boundary conditions at $\varphi = 0$. The solution is assumed in the form

$$w_{tot} = w(\varphi) - w(2\pi - \varphi) + w(2\pi + \varphi) - w(4\pi - \varphi) + w(4\pi + \varphi) + \cdots , \tag{69a}$$

$$u_{tot} = u(\varphi) - u(2\pi - \varphi) + u(2\pi + \varphi) - u(4\pi - \varphi) + u(4\pi + \varphi) - \cdots , \tag{69b}$$

$$v_{tot} = v(\varphi) + v(2\pi - \varphi) + v(2\pi + \varphi) + v(4\pi - \varphi) + v(4\pi + \varphi) + \cdots , \tag{69c}$$

$$w(\varphi) = Ae^{p\varphi} \cos nx, \tag{70a}$$

$$u(\varphi) = Be^{p\varphi} \sin nx, \tag{70b}$$

$$v(\varphi) = Ce^{p\varphi} \cos nx, \tag{70c}$$

where the dependence on $x$ is not shown but is understood, and $A$ and $n$ are assumed real while $B$, $C$, and $p$ may be complex.

Along the generator $\varphi = 0$ an external moment $2M \cos nx$ is applied with $n = m\pi(a/L)$ and $m$ an odd integer. The boundary conditions are

$$w = u = \sigma_\varphi = 0 \qquad \text{when} \qquad \varphi = 0 \tag{71}$$

$$M_{\varphi 0} = M \cos(m\pi ax/L) \tag{72}$$

and $M_{\varphi 0}$ is the internal circumferential moment per unit length of the shell along the cut at $\varphi = 0$.

It follows from (71), (31), (32), and (33) that

$$A_1 + A_3 = 0, \tag{73}$$

$$2A_1K + A_2[(1 + \nu)n + K] - A_4[(1 + \nu)n - K] = 0, \tag{74}$$

$$2A_1\nu K + A_2[(1 + \nu)n + K] - A_4[(1 + \nu)n - \nu K] = 0. \tag{75}$$

Solution of (73) to (75) yields

$$A_4 = A_3 = A_2 = -A_1 . \tag{76}$$

The fourth boundary condition (72) along $\varphi = 0$ gives

$$A_1 = -(1/4\pi)(ML/KD)(1/m). \tag{77}$$

Hence the displacements can be written as

$$w = -(1/4\pi)(ML/KD)(1/m)[e^{-\alpha_1\varphi}(\cos \beta_1\varphi - \sin \beta_1\varphi) - e^{-\alpha_2\varphi}(\cos \beta_2\varphi + \sin \beta_2\varphi)] \cos nx, \tag{78}$$

$$u = (1/8\pi)(ML/K^3D)(1/m)\{e^{-\alpha_1\varphi}\{(1 + \nu)n \cos \beta_1\varphi + [(1 + \nu)n + 2K] \sin \beta_1\varphi\} - e^{-\alpha_2\varphi}\{(1 + \nu)n \cos \beta_2\varphi - [(1 + \nu)n - 2K] \sin \beta_2\varphi\}\} \sin nx, \tag{79}$$

$$\begin{aligned} v = (1/8\pi)(ML/K^3D)(1/mn)\{e^{-\alpha_1\varphi}\{&[(1 + \nu)n(\alpha_1 - \beta_1) + 2\beta_1K] \cos \beta_1\varphi \\ &+ [(1 + \nu)n(\alpha_1 + \beta_1) - 2\alpha_1K] \sin \beta_1\varphi\} \\ &- e^{-\alpha_2\varphi}\{[(1 + \nu)n(\alpha_2 + \beta_2) + 2\beta_2K] \cos \beta_2\varphi \\ &- [(1 + \nu)n(\alpha_2 - \beta_2) + 2\alpha_2K] \sin \beta_2\varphi\}\} \cos nx, \end{aligned} \tag{80}$$

**10. Constant circumferential moment over segment of generator.** When the external circumferential moment is distributed according to the rule

$$\begin{array}{lll} M^*_\varphi/2\delta & \text{when} & \mid x^* \mid \le \delta \\ 0 & \text{when} & (L/2) \ge \mid x^* \mid > \delta \end{array} \tag{81}$$

it can be represented by the Fourier series

$$\sum_{m=1,3,\cdots}^{\infty} M^{*(m)}_\varphi \cos (m\pi ax/L)$$

where the coefficient $M^{*(m)}_\varphi$ is

$$M^{*(m)}_\varphi = (2M^*_\varphi/\pi\delta)(1/m) \sin (m\pi\delta/L). \tag{82}$$

The displacements $w$, $u$, and $v$ can be calculated from (78), (79), and (80) if $M$ is replaced by $(1/2)M^{*(m)}_\varphi$ and the resulting expressions are summed over $m$. Naturally the summation indicated in (69) must also be carried out if the first term alone does not give sufficient accuracy. However, this is seldom the case.

If the coefficient $b_m$ is defined as

$$b_m = -(M^*_\varphi/4\pi^2)(L/DK\delta)(1/m^2) \sin (m\pi\delta/L), \tag{83}$$

the moments and the membrane stresses in the shell can be given in the form

$$M^{(m)}_x/[(-D/a)b_mn \cos nx] = e^{-\alpha_1\varphi}\{[-(1 - \nu)n + 2\nu K] \cos \beta_1\varphi + (1 - \nu)n \sin \beta_1\varphi\} + e^{-\alpha_2\varphi}\{[(1 - \nu)n + 2\nu K] \cos \beta_2\varphi + (1 - \nu)n \sin \beta_2\varphi\}, \tag{84}$$

$$M_\varphi^{(m)}/[(-D/a)b_m n \cos nx] = e^{-\alpha_1\varphi}\{[(1-\nu)n + 2K]\cos\beta_1\varphi - (1-\nu)n\sin\beta_1\varphi\}$$
$$+ e^{-\alpha_2\varphi}\{[-(1-\nu)n + 2K]\cos\beta_2\varphi - (1-\nu)n\sin\beta_2\varphi\}, \qquad (85)$$

$$M_{x\varphi}^{(m)}/[(-D/a)(1-\nu)b_m n\sin nx] = e^{-\alpha_1\varphi}\{-(\alpha_1+\beta_1)\cos\beta_1\varphi + (\alpha_1-\beta_1)\sin\beta_1\varphi\}$$
$$+ e^{-\alpha_2\varphi}\{(\alpha_2-\beta_2)\cos\beta_2\varphi + (\alpha_2+\beta_2)\sin\beta_2\varphi\}, \qquad (86)$$

$$\sigma_x^{(m)}/[(E/2K^2)b_m n\cos nx] = -e^{-\alpha_1\varphi}\{n\cos\beta_1\varphi + (n+2K)\sin\beta_1\varphi\}$$
$$+ e^{-\alpha_2\varphi}\{n\cos\beta_2\varphi - (n-2K)\sin\beta_2\varphi\}, \qquad (87)$$

$$\sigma_\varphi^{(m)}/[E/2K^2)b_m n^2\cos nx] = e^{-\alpha_1\varphi}\{\cos\beta_1\varphi + \sin\beta_1\varphi\}$$
$$+ e^{-\alpha_2\varphi}\{-\cos\beta_2\varphi + \sin\beta_2\varphi\}, \qquad (88)$$

$$\tau_{x\varphi}^{(m)}/[(E/2K^2)b_m n\sin nx] = e^{-\alpha_1\varphi}\{(\alpha_1-\beta_1)\cos\beta_1\varphi + (\alpha_1+\beta_1)\sin\beta_1\varphi\}$$
$$+ e^{-\alpha_2\varphi}\{-(\alpha_2+\beta_2)\cos\beta_2\varphi + (\alpha_2-\beta_2)\sin\beta_2\varphi\}. \qquad (89)$$

When the moment is concentrated at $x = 0$, the distance $\delta$ approaches zero. In that case

$$M_\varphi^{(m)} = (2M_\varphi^*/L) \qquad (90)$$

and

$$b_m = -(M_\varphi^*/4\pi DK)(1/m). \qquad (91)$$

**11. Convergence.** When $\delta$ is finite in the series representing the loads, all the displacements, moments, and stresses converge. This is true even when $\delta = 0$, $x \neq 0$, and the load is a concentrated radial force $P^*$. In this case, when $m$ is sufficiently large, the quantities of practical interest behave at $\varphi = 0$ in the following manner:

$$w \sim (1/m^3)\cos(m\pi ax/L), \qquad w_{,x} \sim (1/m^2)\sin(m\pi ax/L),$$
$$M_x \sim (1/m)\cos(m\pi ax/L), \qquad M_\varphi \sim (1/m)\cos(m\pi ax/L), \qquad (92)$$
$$\sigma_x \sim (1/m^3)\cos(m\pi ax/L), \qquad \sigma_\varphi \sim (1/m^3)\cos(m\pi ax/L),$$
$$\tau_{x\varphi} = M_{x\varphi} = 0.$$

When $\varphi = 0$ and $x = 0$, the series for $M_x$ and $M_\varphi$ diverge, but the remaining series converge.

When the load is a concentrated longitudinal moment at $x = 0$, $\varphi = 0$, the behavior at $\varphi = 0$ for large values of $m$ is:

$$w \sim (1/m^2)\sin(m\pi ax/L), \qquad w_{,x} \sim (1/m)\cos(m\pi ax/L),$$
$$M_x \sim \sin(m\pi ax/L), \qquad M_\varphi \sim \sin(m\pi ax/L), \qquad (93)$$
$$\sigma_x \sim (1/m^2)\sin(m\pi ax/L), \qquad \sigma_\varphi \sim (1/m^2)\sin(m\pi ax/L),$$
$$\tau_{x\varphi} = M_{x\varphi} = 0$$

The displacements and the membrane stresses again converge, but the moments $M_x$ and $M_\varphi$ are given by divergent series representing zero when $x \neq 0$ and an indefinitely large quantity when $x = 0$. The series converge for all values of $\varphi \neq 0$.

Finally, when the load is a concentrated circumferential moment at $\varphi = 0$, $x = 0$, one obtains at $\varphi = 0$ for sufficiently large $m$:

$$w = w_{,x} = \sigma_x = \sigma_\varphi = 0,$$

$$M_x \sim \cos(m\pi ax/L), \qquad M_\varphi \sim \cos(m\pi ax/L), \tag{94}$$

$$M_{x\varphi} \sim \sin(m\pi ax/L), \qquad \tau_{x\varphi} \sim (1/m^2)\sin(m\pi ax/L).$$

Here again the series for the moments diverge in consequence of the representation of the load by a divergent series. The series for the stresses converge, and all the quantities are represented by convergent series when $\varphi \neq 0$.

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# PLASTIC FLOW IN A DEEPLY NOTCHED BAR WITH SEMI-CIRCULAR ROOT*

BY

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**Summary.** The unsteady motion problem of a circular-notched bar pulled in tension in plane strain is considered. The theory of perfectly plastic solids is used. Large strains are analyzed so that the material can also be considered as plastic-rigid. The basic equations governing stress and velocity are integrated independently in the characteristic plane. The results are used to construct the boundary change in a step-by-step manner. The problem is greatly simplified because at each step the new free boundary of the plastic region can be approximated by a circle. The final shape of the boundary of an initially semi-circular notch is presented when plastic flow has reduced the initial connection at the root to a line contact between the shanks.

**1. Introduction.** We consider the plastic flow in a deeply notched bar under tension with semi-circular root. For a deep notch, the plastic flow is localized in the vicinity of the root of the notch and the parts remaining elastic prevent appreciable lateral contraction, thus allowing us to consider the plastic flow to be in plane strain. The present paper follows a series of other papers on the deformation of notched bars under tension with $V$-shaped roots [1]** and rectangular roots [2]. The former is a quasi-steady case in which the configuration maintains geometrical similarity. The latter is an unsteady case in which the analysis can be readily made by building up some known solutions of slip line fields. The present treatment is another attempt at an unsteady case. Furthermore, since actual specimens have circular fillets at the corners, the result can also be extended to interpret such tension experiments. Thus this solution may help to examine more completely the singularities created by sharp corners in the previous cases.

The present paper limits itself to the case when the plastic region extends only to the circular part of the notch, Fig. 1 where $C_0'BC_0$ is the circular part and $C_0D$, $C_0'D'$ are the linear parts of the free boundary. The case when the plastic region extends to

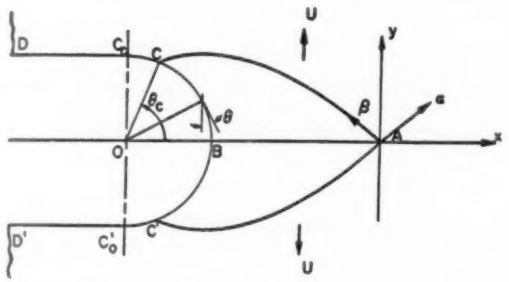


FIG. 1.

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**Numbers in square brackets refer to the references at the end of the paper.

the linear part of the notch and the case of a circular fillet will be considered in later work.

We shall use plastic rigid theory. It is valid here as well as in most metal forming problems, since plastic flow needs to be analyzed in the regions of large flow only. We shall also consider the material to be perfectly plastic, i.e., plastic flow occurs at a constant stress limit. It is not equivalent to neglecting work-hardening, but rather to averaging its effect over the field of flow. The basic theory of such analysis has been fully discussed in the recent literature [3], [4], [5]. Accordingly, only a brief résumé of the final equations is given below.

**2. Equations governing plastic flow in plane strain.** In the plastic region the stress has to satisfy the yield condition and the equilibrium equations. In plane strain these three equations lead to a pair of first order hyperbolic equations having two orthogonal sets of real characteristics commonly called slip lines. The equations expressed in terms of these lines as coordinates (called the canonical form in the theory of partial differential

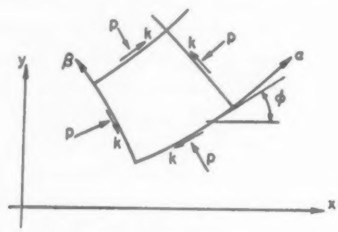


Fig. 2.

equations) take on a particularly simple form. Referring to the notation of Fig. 2, the equilibrium equations become

$$\begin{aligned} p + 2k\varphi &= \text{constant along an } \alpha\text{-line}, \\ p - 2k\varphi &= \text{constant along a } \beta\text{-line}, \end{aligned} \tag{1}$$

and so

$$\frac{\partial^2 \varphi}{\partial \alpha \, \partial \beta} = 0. \tag{2}$$

Equation (2) governing the slip line field can also be expressed in terms of the radii of curvature $R$, $S$ of the $\alpha$ and $\beta$ lines respectively, giving the alternative relations

$$\begin{aligned} dS + R\, d\varphi &= 0 \text{ along an } \alpha\text{-line}, \\ dR - S\, d\varphi &= 0 \text{ along a } \beta\text{-line}. \end{aligned} \tag{3}$$

The condition of incompressibility and the relation between stress and strain-rate lead to a similar set of equations for velocity components. They have the same characteristics. If $u$ and $v$ denote the velocity components along the $\alpha$ and $\beta$-lines, respectively, we have

$$\begin{aligned} du - v\, d\varphi &= 0 \text{ along an } \alpha\text{-line} \\ dv + u\, d\varphi &= 0 \text{ along a } \beta\text{-line}. \end{aligned} \tag{4}$$

We shall use the convention that $R$, $S$ are positive in the directions of increasing $\beta$ and $\alpha$, respectively. If the curvatures of the $\alpha$, $\beta$ net are as shown in Fig. 2, $R$ will be positive and $S$ negative, then

$$\varphi = \alpha + \beta + \text{constant}, \tag{5}$$

$$\frac{p}{k} = -\alpha + \beta + \text{constant}, \tag{6}$$

$$\frac{\partial S}{\partial \alpha} = -R, \qquad \frac{\partial R}{\partial \beta} = S. \tag{7}$$

We then have for (3) and (4)

$$\text{(a)} \quad \frac{\partial^2 R}{\partial \alpha\, \partial \beta} = -R, \qquad \text{(b)} \quad \frac{\partial^2 S}{\partial \alpha\, \partial \beta} = -S, \tag{8}$$

$$\frac{\partial^2 u}{\partial \alpha\, \partial \beta} = -u, \qquad \frac{\partial^2 v}{\partial \alpha\, \partial \beta} = -v, \tag{9}$$

In the following, (8) will be referred to as the stress equations and (9) as the velocity equations. They are all of the form of the telegraph equation [6].

The transformation from the characteristic plane to the physical plane is achieved by integrating the following relations:

$$\begin{aligned} \frac{\partial x}{\partial \alpha} &= R \cos \varphi, & \frac{\partial x}{\partial \beta} &= S \sin \varphi, \\ \frac{\partial y}{\partial \alpha} &= R \sin \varphi, & \frac{\partial y}{\partial \beta} &= -S \cos \varphi. \end{aligned} \tag{10}$$

**3. Analytical solutions.** Finding the plastic flow inside the plastic region requires the solutions of equations (8) and (9) with their respective boundary conditions. The solutions of (8) will give the shape of the slip lines at any instant. Since the slip lines are the trajectories directed along the plane of maximum shear stress, we can find the stress distribution throughout the plastic region. The solutions of (9) will in turn give the strain-rate distribution at any instant. The two sets of equations can be solved separately in the characteristic plane; thereafter they are transformed back to the physical plane.

When the unsteady motion of the initially semi-circular notched root is investigated, it is found that the free-surface of the plastic region maintains very closely the form of a circular arc. To carry out the complete analysis at all stages in the deformation, it is therefore only necessary to evaluate the stress and velocity fields for a free boundary having circular form. This basic problem is considered in detail below.

First of all, we shall determine the constants in equations (5) and (6). By symmetry we may consider one half of the bar. Let the free surface of the plastic region subtend an angle $2\theta_c$ of a circular arc, and take the origin at the center of the bar, Fig. 1. Equation (5) becomes

$$\varphi = \alpha + \beta + \frac{\pi}{4}. \tag{11}$$

On the free boundary $p = -k$. This determines the constant in (6), giving

$$\frac{p}{2k} = -\alpha + \beta - \theta_c - \frac{1}{2}. \tag{12}$$

The free boundary $BC$ therefore has the equation

$$-\alpha + \beta = \theta_c . \tag{13}$$

On the boundary we also have the relation,

$$\alpha + \beta = \theta. \tag{14}$$

Next, we look for the solution to the stress Eqs. (8). Because of symmetry it is necessary to solve for only one of the terms $R$ or $S$. Hence, for $S$

$$\frac{\partial^2 S}{\partial\alpha\,\partial\beta} + S = 0. \tag{15}$$

The boundary conditions on $BC$ are:

$$R = -S = +2^{1/2}a.$$

Using (7), we have

$$\frac{\partial S}{\partial \alpha} = -2^{1/2}a.$$

Making use of the fact that $BC$ is a circular arc, we find

$$\frac{\partial S}{\partial \beta} = 2^{1/2}a.$$

The boundary conditions are therefore,

$$S = -2^{1/2}a, \qquad \frac{\partial S}{\partial \alpha} = -2^{1/2}a, \qquad \frac{\partial S}{\partial \beta} = 2^{1/2}a. \tag{16}$$

This is a Cauchy problem since two boundary conditions are given on a curve which is nowhere tangent to a characteristic. The solution is obtained by Riemann's method [6]. The Riemann's function in this case is $J_0[2(\xi - \alpha)^{1/2}(\eta - \beta)^{1/2}]$.

The solution is obtained by considering the general formulation of Green's theorems which in this case reduces to the following

$$\oint \left\{ J_0 \frac{\partial S}{\partial \beta}\, d\beta + S \frac{\partial J_0}{\partial \alpha}\, d\alpha \right\} = 0.$$

With the path $PTQ$ as indicated in Fig. 3, we have,

$$\begin{aligned} S_p(\xi, \eta) &= S_Q - \int_{TQ} \left\{ J_0 \frac{\partial S}{\partial \beta}\, d\beta + S \frac{\partial J_0}{\partial \alpha}\, d\alpha \right\} \\ &= -2^{1/2}a - \int_{\eta}^{\xi+\theta_c} J_0[2i(\xi - \beta + \theta_c)^{1/2}(\beta - \eta)^{1/2}]2^{1/2}a\, d\beta \\ &\qquad - \int_{\eta-\theta_c}^{\xi} J_0'\left[ i\left(\frac{2(\alpha + \theta_c - \eta)}{\xi - \eta}\right)^{1/2} \right] a\, d\alpha \\ &= -2^{1/2}a\{1 + P_I + P_{II}\} \\ &= -2^{1/2}a\left\{1 + 2\sum_{m=0}^{\infty} (-)^m \frac{B^{m+1}}{m!(2m+1)}\,[I_m(2B) + I_{m+1}(2B)]\right\}, \end{aligned} \tag{17}$$

where $2B = \theta_c - \eta + \xi$. The last step is obtained through the help of Sonine's integral representation of Bessel's function [7].

We know, however, from consideration of symmetry that the slip lines form logarithmic spirals with radii of curvature

$$S = -2^{1/2}r = -2^{1/2}ae^{\theta_c - \theta_p} = -2^{1/2}ae^{2B}. \tag{18}$$

The numerical identification of (17) and (18) is readily established, but the reduction of (17) to (18) through Bessel function recurrence relations appears to be quite complicated.

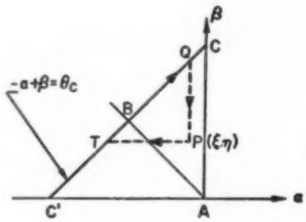


Fig. 3.

By symmetry,

$$R(\xi, \eta) = -S(-\eta, -\xi). \tag{18a}$$

The pressure distribution in the plastic region can be found easily, being especially simple across the line of symmetry $AB$. On $AB$, $-\alpha = \beta$ so that

$$\frac{p}{2k} = -2\alpha - \theta_c - \frac{1}{2},$$

$$\sigma_y = -p + k \qquad \text{(tension)} \qquad = 2k(1 + \theta_c + 2\alpha).$$

With

$$x = a(e^{\theta_c} - e^{\theta_p}),$$

$$\theta_p = \alpha + \beta = 2\alpha + \theta_c\,,$$

$$\theta_p = \log\left(e^{\theta_c} - \frac{x}{a}\right).$$

We obtain (see Fig. 4)

$$\frac{p}{2k} = -\log\left(e^{\theta_c} - \frac{x}{a}\right) - \frac{1}{2},$$

$$\sigma_y = 2k\left[1 + \log\left(e^{\theta_c} - \frac{x}{a}\right)\right]. \tag{19}$$

These relations are plotted in Fig. 4.

Next, we look at the solution to the velocity equations (9). By symmetry we may consider $u$ only. Thus, we have

$$\frac{\partial^2 u}{\partial \alpha\, \partial \beta} + u = 0 \tag{20}$$

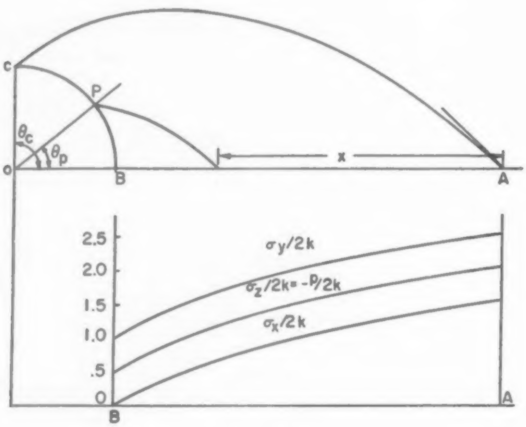


Fig. 4.

With the boundary conditions

$$u = \sin \varphi \qquad \text{along } AC,$$

$$v = -\cos \varphi = \frac{\partial u}{\partial \alpha} \qquad \text{along } AC', \tag{21a}$$

where $U$ has been taken as the unit of velocities.

The last relation is obtained from Eq. (4). At the point $A$, we see that there is a jump in $u$ across $AC'$ of magnitude $2^{1/2}$. Integrating the last relation we find that, along $AC'$,

$$u = -(\sin \varphi - 2^{1/2}). \tag{21b}$$

Note that the boundary conditions on the plastic rigid boundaries $AC$ and $AC'$ depend only on $\varphi$ and not on the exact shape of the curve. Therefore, the result can be applied for any shape of plastic rigid boundary, the particular discontinuity of $2^{1/2}$ in $u$ requiring the body to be symmetric at least about one of the $x$ and $y$ axes.

The problem of solving (20) and (21) is a problem of the Riemann type since we are given one condition on each of the two characteristics. The Riemann's function is again $J_0[2(\xi - \alpha)^{1/2}(\eta - \beta)^{1/2}]$. Using

$$\oint \left[ J_0 \frac{\partial u}{\partial \beta}\, d\beta + u \frac{\partial J_0}{\partial \alpha}\, d\alpha \right] = 0$$

with the path $PQAT$ as indicated in Fig. 5, we have

$$u_P(\xi,\eta) = u(\xi, 0) - \int_{QA} J_0 \frac{\partial u}{\partial \beta}\, d\beta - \int_{AT} u \frac{\partial J_0}{\partial \alpha}\, d\alpha$$

$$= 2^{1/2} - \sin\left(\xi + \frac{\pi}{4}\right) + \int_0^\eta J_0[2\xi^{1/2}(\eta - \beta)^{1/2}] \cos\left(\beta + \frac{\pi}{4}\right) d\beta$$

$$+ \int_\xi^0 \left[2^{1/2} - \sin\left(\alpha + \frac{\pi}{4}\right)\right] \frac{\partial J_0[2\eta^{1/2}(\xi - \alpha)^{1/2}]}{\partial \alpha}\, d\alpha.$$

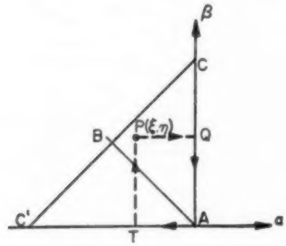


FIG. 5.

Integrating the second integral by parts and changing the variables, we obtain

$$u_P(\xi,\eta) = 2^{-1/2} I_0[2(-\xi\eta)^{1/2}] + \int_0^\eta I_0[2(-\xi z)^{1/2}] \cos\left(\eta + \frac{\pi}{4} - z\right) dz$$

$$+ \int_0^{-\xi} I_0[2(\eta z)^{1/2}] \cos\left(\xi + \frac{\pi}{4} + z\right) dz$$

$$= 2^{-1/2} I_0[2(-\xi\eta)^{1/2}][1 + \eta - \xi] + 2^{-1/2} \sum_{p=1}^{\infty} I_{2p}[2(-\xi\eta)^{1/2}]$$

$$\cdot \left\{\left(\frac{\eta}{\xi}\right)^p \left[\frac{\eta}{2p+1} + 1 - \frac{\xi}{2p-1}\right] - \left(\frac{\xi}{\eta}\right)^p \left[\frac{\xi}{2p+1} + 1 - \frac{\eta}{2p-1}\right]\right\}. \tag{22}$$

The last step is obtained through the help of Sonine's integral representation of Bessel's function [7]. The convergence of the series in the range considered is very good. By symmetry we have

$$v(\xi,\eta) = -u(-\eta, -\xi). \tag{22a}$$

In the step-by-step construction of the boundary, the following expressions are useful. On the boundary

$$\xi = \tfrac{1}{2}(\theta - \theta_c), \qquad \eta = \tfrac{1}{2}(\theta + \theta_c)$$

Hence, the radial velocity

$$U_r = 2^{-1/2}(u - v) = (1 + \theta_c) I_0[(\theta_c^2 - \theta^2)^{1/2}]$$

$$- \sum_{p=1}^{\infty} I_{2p}[(\theta_c^2 - \theta^2)^{1/2}] \left[\left(\frac{\theta + \theta_c}{\theta - \theta_c}\right)^p \left(\frac{\theta - 2p\theta_c}{4p^2 - 1}\right) - \left(\frac{\theta - \theta_c}{\theta + \theta_c}\right)^p \left(\frac{\theta - 2p\theta_c}{4p^2 - 1}\right)\right] \tag{23}$$

and the tangential velocity

$$U_t = 2^{-1/2}(u + v) = \sum_{p=1}^{\infty} I_{2p}[(\theta_c^2 - \theta^2)^{1/2}]\left[\left(\frac{\theta + \theta_c}{\theta - \theta_c}\right)^p - \left(\frac{\theta - \theta_c}{\theta + \theta_c}\right)^p\right] \tag{24}$$

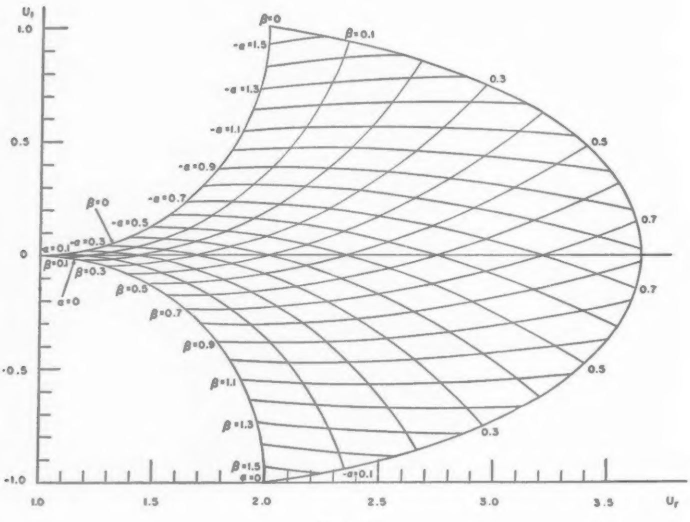


FIG. 6.

These quantities are plotted in Fig. 6 for $\theta_c$ ranging from 0 to $\pi/2$. At the point $B$,

$$(U_t)_B = 0,$$

$$(U_r)_B = 2^{1/2}u_B = I_0(\theta_c) + 2\sum_{p=0}^{\infty}(-)^p I_{2p+1}(\theta_c).$$

Since $\sinh z = 2\sum_{p=0}^{\infty} I_{2p+1}(z)$ (being transformed from the expression for $\sin z$ [7]), we have

$$2^{1/2}u_B = I_0(\theta_c) + \sinh\theta_c - 4\sum_{p=1}^{\infty} I_{4p-1}(\theta_c).$$

The convergence of the last expression is particularly rapid; for 4 significant figures one term in the summation is enough, i.e.,

$$2^{1/2}u_B = I_0(\theta_c) + \sinh\theta_c - 4I_3(\theta_c) \tag{25}$$

The transformation functions are obtained in the following manner. With

$$S = -2^{1/2}ae^{\theta_c-\beta+\alpha}, \qquad R = 2^{1/2}ae^{\theta_c-\beta+\alpha}$$

we can integrate Eq. (10) obtaining

$$x = 2^{1/2}a\int e^{\theta_c-\beta+\alpha}\cos\left(\alpha + \beta + \frac{\pi}{4}\right)d\alpha + f(\beta)$$

or

$$= -2^{1/2}a \int e^{\theta_c - \beta + \alpha} \sin\left(\alpha + \beta + \frac{\pi}{4}\right) d\beta + g(\alpha)$$

or

$$x = ae^{\theta_c - \beta + \alpha} \cos(\alpha + \beta) + C_1 .$$

At $\alpha = \beta = 0$, $x = 0$ so $C_1 = -ae^{\theta_c}$. Therefore,

$$x = ae^{\theta_c}[e^{\alpha - \beta} \cos(\alpha + \beta) - 1].$$

Similarly,

$$y = a[e^{\theta_c - \beta + \alpha} \sin(\alpha + \beta)]. \tag{26}$$

They appear to be quite simple, however, with $\alpha$, $\beta$ and $\theta_c$ all varying with time, the transformation functions for a generic $t$ are very complicated.

**4. Approximate solutions.** Since the analytical solutions above are not in closed form, the dependence of $\theta_c$, $\alpha$, $\beta$ on $t$ becomes very complicated and it is much simpler to analyze the unsteady motion by approximate means. If we represent a boundary curve in intrinsic coordinates $R$ and $S$ and if we know the normal and tangential velocity components along this curve, then the subsequent shape of the boundary will depend only on

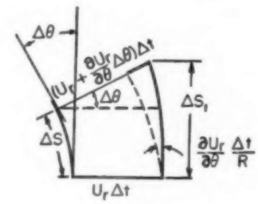


FIG. 7.

the normal velocity component. Referring to Fig. 7, we have, after the time $\Delta t$,

$$R_1 = R + \Delta t\left[U_r - \frac{1}{R}\frac{\partial}{\partial\theta}(R)U_r' + U_r''\right] + O(\Delta t^2), \tag{27}$$

where we denote differentiation with respect to $\theta$ by a prime.

For an originally circular boundary, $\partial R/\partial\theta = 0$; $U_r + U_r''$ was numerically computed for $\theta_c = \pi/4$. The result shows that if $\Delta t$ is taken as 1, the deviation of the new boundary from a circle with the same original center is within $\pm 2.0\%$ and if $\Delta t$ is taken as 0.1, the deviation is within $\pm 0.4\%$. Furthermore, we can change the position of the center of the circle with the resulting error of 0.33% for $\Delta t = 1$ and 0.14% for $\Delta t = 0.1$. In the following step-by-step construction, a circle is drawn through the new positions of $B$ and $C$ with the center remaining on the $x$-axis. The validity of approximating the new boundary by such a circular arc is checked all along the step-by-step process for $\theta_c$ ranging from $\pi/2$ and zero. The maximum error is about 0.2%. It is considered to lie within our error of graphical construction.

**5. Step-by-step process.** After a small time increment $\Delta t$, an originally circular

arc boundary $BC$ will take the form $B_2C_2$, Fig. 8. We shall pass a circle through $B_2$ and $C_2$ with center $O_2$ on the $x$-axis. Hence,

$$(r + \Delta r)^2 = \left[a \cos \theta_c + \left(\frac{dx}{dt}\right)_c \Delta t - \Delta x_0\right]^2 + \left[a \sin \theta_c + \left(\frac{dy}{dt}\right)_c\right]^2, \tag{28}$$

$$r + \Delta r + \Delta x_0 = r + 2^{1/2} u_B \, \Delta t. \tag{29}$$

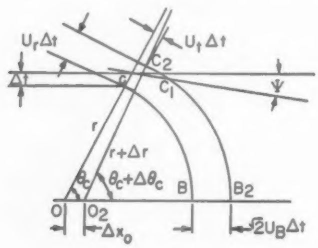


FIG. 8.

For the values at $C$, we can make use of the boundary conditions. Then,

$$\left(\frac{dx}{dt}\right)_c = u_c \sin\left(\frac{\pi}{4} - \theta_c\right) - v_c \cos\left(\frac{\pi}{4} - \theta_c\right)$$

$$= \cos \theta_c + \sin \theta_c ,$$

$$\left(\frac{dy}{dt}\right)_c = 1 + \sin \theta_c - \cos \theta_c .$$

Solving (28) and (29) for $\Delta x_0$ and $\Delta r$, we obtain

$$\Delta x_0 = \frac{2^{1/2} u_B - 1 - \sin \theta_c}{1 - \cos \theta_c} \Delta t + O(\Delta t^2), \tag{30}$$

$$\Delta r = \frac{-2^{1/2} u_B \cos \theta_c + 1 + \sin \theta_c}{1 - \cos \theta_c} \Delta t + O(\Delta t^2), \tag{31}$$

$$\theta_c + \Delta\theta_c = \log \frac{re^{\theta_c} - \Delta x_0}{r + \Delta r}$$

$$= \theta_c + \frac{\Delta x_0}{r}(1 - e^{-\theta_c}) - \frac{2^{1/2} u_B}{r} \Delta t + O(\Delta t^2). \tag{32}$$

Since these algebraic equations are expressed in terms of $\theta_c$ and $2^{1/2}u_B$ only, where $2^{1/2}u_B$ is represented by Eq. (25), it is easier to work with these equations than to interpolate the results from Fig. 6.

An example for an initial $\theta_c = \pi/2$ is carried out. The free boundary is found after a small time increment $\Delta t$ by using eqns. (30), (31) and (32). The first step is to replace $\theta_c$ by $\pi/2$, put $2^{1/2}u_B(\pi/2) = 3.644$ in these eqns. and find the position of the new center and the values of the new radius and of $\theta_{c1}$. For this new value of $\theta_{c1}$ we can calculate the corresponding $2^{1/2}u_B$. The new values of $\theta_c$ and $2^{1/2}u_B$ are then substituted in the three equations to find out the next $\theta_c$ and so forth. The magnitude of the time incre-

ment is chosen to provide as rapid a procedure as possible in conjunction with satisfactory accuracy. In the present example $\Delta t = 0.050$ with $r = 1$ at that instant. The accuracy of such a step is checked by taking $\Delta t = 0.025$ for two steps and comparing the result with that of one step of $\Delta t = 0.050$. The resulting points are indistinguishable. However, similar comparison between two steps of $\Delta t = 0.05$ and one step of $\Delta t = 0.10$ shows that $\Delta t = 0.10$ yields considerable error. Therefore, the step of $\Delta t = 0.05$ is taken and the resulting error is no greater than the unavoidable graphical errors. The

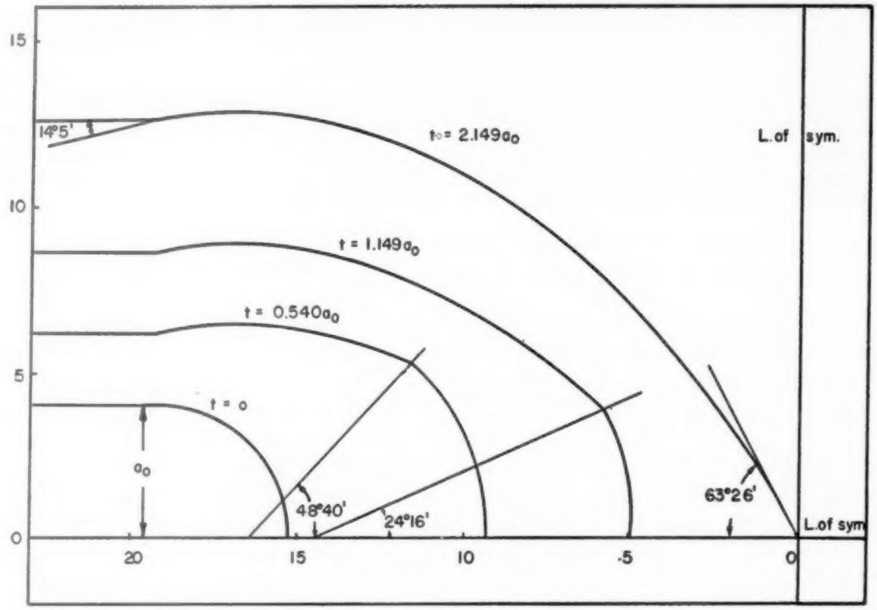


FIG. 9.

final shape along with two intermediate steps are shown in Fig. 9. From Fig. 8 the initial discontinuity in slope is calculated as follows

$$\tan \Psi = \frac{(r + \Delta r) \sin (\theta_c + \Delta \theta_c) - r \sin \theta_c - \Delta t}{\Delta x_0 + (r + \Delta r) \cos (\theta_c + \Delta \theta_c) - r \cos \theta_c}. \tag{33}$$

For $\theta_c = \pi/2$ and $\theta_c + \Delta\theta_c \doteq \pi/2$, $\sin (\theta_c + \Delta\theta_c) \doteq 1$ and $\cos (\theta_c + \Delta\theta_c) \doteq -\Delta\theta_c$

$$\tan \Psi = \frac{\Delta r - \Delta t}{\Delta x_0 - r\Delta\theta_c}$$

$$= \frac{2.644\Delta t - 1.644\Delta t}{(1.644e^{-\pi/2} + 3.644)\Delta t} = \frac{1}{3.986} = .2509,$$

$$\Psi = 14° \, 5'.$$

Note that the boundary forms a recess at the point $C$ which is also the case for a $V$-notched bar. The final angle of 63° 26′ can be found in a similar manner.

**6. Remarks.** We note that the problem is conveniently separated into two mutually independent ones, viz., stress and velocity solutions. These are then combined to give the solution to the over-all problem. However, the combination is complicated by the fact that both $\alpha$ and $\beta$ are functions of time. Complicated integral expressions arise and so step-by-step method is adopted. Fortunately, the approximation of the new boundary by a circular arc is good, making the computation reasonably simple.

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# ON THE STABILITY OF SOME FLOWS OF AN IDEAL FLUID WITH FREE SURFACES*

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**1. Introduction.** Steady state plane flows of an incompressible inviscid fluid with free surfaces were originally studied by Helmholtz [2] and Kirchoff [3], and have since been thoroughly reported in the literature. Their work was an attempt to improve the classical solutions of flow around sharp corners which are physically unacceptable because they involve infinite velocities at the corners. Helmholtz and Kirchoff reasoned that as the velocity becomes large, the pressure in the fluid decreases to the value at which the fluid goes over into the vapor state. This gives rise to a so-called cavitated region bounded by a "free surface" over which the pressure is assumed to be maintained constant and uniform.

One also deals with steady free surface flows in the case of jets flowing in an ambient constant pressure atmosphere. Several examples of such problems are treated by Milne-Thompson [4].

Very little work has been done in the past concerning time dependent flows with free surfaces. Lord Kelvin [5] discussed the vibrations of a hollow columnar vortex flow. Some unsteady free surface flows under the influence of external forces, such as gravity waves in water, are discussed in Lamb [6] and the investigation there is extended to include the effects of surface tension and viscosity.

Recently Ablow and Hayes [1] developed a theory of the small perturbations of the two-dimensional flow of a perfect fluid in the presence of a free surface without external forces. They then used their theory to study two specific problems, namely the flow around a hollow vortex and the flow through a Borda mouthpiece.

The present investigation will concern itself with an extension of the work of Ablow and Hayes to some free surface flows of jets as well as to a number of generalizations of problems treated in [1]. Our primary concern will be to obtain information concerning the stability of these flows.

**2. Resume of basic theory.** The basic theory underlying the methods used in this report has been discussed in detail in the work of Ablow and Hayes [1]. For the sake of convenience however, a brief outline of the important results will be given below.

*A. Assumptions.* We shall be dealing with perturbations of steady state flows which do not fill the entire plane. They can be conveniently divided into three categories: (1) flows which are cavitated due to the fact that there is a minimum pressure the fluid can sustain; (2) jet type flows; (3) flows which are a combination of (1) and (2). In all cases there exist in the steady flow free surfaces along which the pressure remains constant and uniform.

The fluid is assumed homogeneous, incompressible and inviscid. Both the steady and perturbed states are assumed to be irrotational and two-dimensional.

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All quantities are written in non-dimensional form through the use of a characteristic length, pressure and velocity in such a manner as to make the steady state velocity along the free surface of unit magnitude.

Under the assumptions made the flows must satisfy Bernoulli's equation in the form

$$\varphi + \rho q^2 + \rho\dot{\varphi} = C(t), \tag{2.1}$$

where $p$ is the pressure, $\rho$ the density, $q$ the velocity, $\varphi$ the velocity potential and $C(t)$ is a function of time alone. The dot indicates partial differentiation with respect to time.

Furthermore, we can introduce a complex potential $f$ and a complex velocity $w$ such that

$$w = \frac{df}{dz}, \tag{2.2}$$

where $w = u - iv$ and $u$ and $v$ are the Cartesian velocity components in the $z = x + iy$ plane.

*B. Basic flow.* The basic steady flow satisfies the steady form of (2.1)

$$p_0 + \tfrac{1}{2}\rho q_0^2 = \text{const.}, \tag{2.3}$$

where the use of a zero subscript denotes the basic steady flow. Since

$$w_0 = \frac{df_0}{dz_0} \quad \text{and} \quad q_0 = |\, w_0 \,|,$$

we can write (2.3) as

$$p_0 + \tfrac{1}{2}\rho w_0 w_0^* = \text{const.},$$

where the star indicates the operation of taking the complex conjugate*.

*C. Perturbation relations.* We shall now give the steady state basic flow a small perturbation in terms of a small real parameter $\epsilon$ in the form

$$z = z_0 + \epsilon z_1(z_0\,,\,t),$$

$$f(z,\,t) = f_0(z_0) + \epsilon f_1(z_0\,,\,t),$$

$$w(z,\,t) = w_0(z_0) + \epsilon w_1(z_0\,,\,t),$$

$$p(z,\,t) = p_0(z_0) + \epsilon p_1(z_0\,,\,t),$$

where $f$ and $w$ are analytic functions of $z_0$. All subsequent relations will be linearized by neglecting terms of order $\epsilon^2$ and higher. Hence all perturbations are small perturbations in that they are correct only to first order in $\epsilon$. It is convenient to perturb the independent variable $z_0$, although the perturbations in $f$, $w$ and $p$ are given in terms of the fixed point $z_0$.

The perturbations given above are not independent since we can derive the following relations from (2.1) and (2.2)

$$w_0^2 z_1' + w_1 = w_0 f_1', \tag{2.4}$$

$$p_1 + \mathrm{Re}[w_1 w_0^* + \dot{f}_1 - w_0 \dot{z}_1] = 0, \tag{2.5}$$

*The star is used instead of the usual bar, for typographical reasons.

where the prime indicates partial differentiation with respect to $f_0$, and Re denotes the real part. Thus we see that only two of the four perturbations are independent.

We note that, when properly chosen, two different sets of perturbations, e.g., $(z_3, f_3)$ and $(z_4, f_4)$ may represent the same physical perturbation. Their difference, namely $z_I = z_3 - z_4$, $f_I = f_3 - f_4$, will leave the flow unchanged and the perturbation $(z_I, f_I)$ will be called an invariant perturbation.

We define a stationary perturbation $(z_2, f_2)$ to be one in which any given physical perturbation is evaluated at a fixed point $z_0$ of the basic flow, i.e., one for which the space variable is not perturbed ($z_2 \equiv 0$). We can now find, corresponding to a given perturbation $(z_1, f_1)$, a unique stationary form by superposing on $(z_1, f_1)$ the invariant perturbation $z_I = -z_1$.

Using (2.4) and (2.5) we can derive the following relations between the stationary perturbations $f_2$, $w_2$ and $p_2$:

$$w_2 = w_0 f_2', \tag{2.6}$$

$$p_2 = -\rho \operatorname{Re} [w_0 w_0^* f_2' + \dot{f_2}]. \tag{2.7}$$

In this formulation there is only one independent perturbation quantity, say $f_2$, restricted only by the condition that it be admissible under the boundary conditions of the problem.

In subsequent work, for the sake of compactness, we shall not change the name of a function after a change of independent variable, e.g., we shall write

$$f(z_0) = f[w_0(z_0)] = f(w_0).$$

*D. Free surface condition.* There are two conditions that must hold on the free surface. First, the free surface pressure remains constant, and second, a particle originally on the free surface remains on the free surface in the perturbed state. From (2.5) we see that we can satisfy the first condition by demanding that

$$\operatorname{Re} [f_2' + \dot{f_2} + w_0' z_1] = 0 \tag{2.8}$$

where we have used the relation between $f_1$ and its stationary form $f_2$

$$f_2 = f_1 - w_0 z_1 .$$

The second condition can be shown to imply that

$$\operatorname{Im} [f_2' + (w_0 z_1)' + (w_0 z_1)^{\cdot}] = 0 \tag{2.9}$$

where Im denotes the imaginary part.

We can satisfy (2.8) identically if we set the expression in the bracket equal to $i\chi(z_0, t)$, where the function $\chi$ is real on the free surface and otherwise arbitrary. Solving for $z_1$ to obtain the perturbation in the space co-ordinate which takes the basic flow free surface into the perturbed free surface, we have

$$z_1 = \frac{1}{w_0'} \{i\chi - D[f_2]\} \tag{2.10}$$

where the operator $D[\ \ ] = \partial[\ \ ]/\partial f_0 + \partial[\ \ ]/\partial t$. Now substituting for $z_1$ from (2.10) in (2.9) we find, after some reduction, that the free surface boundary condition is

$$\operatorname{Im} \{D[f_2 - \omega D[f_2]] - \dot{f_2}\} = 0 \tag{2.11}$$

where $\omega = w_0/w_0'$.

Adopting the notation

$$H \equiv L[f_2] \equiv D[f_2 - \omega D[f_2]] - f_2^{\cdot} \tag{2.12}$$

(2.11) becomes with $w_0$ as the independent variable

$$H(w_0) = (H(w_0))^* \qquad \text{on} \qquad w_0 w_0^* = 1. \tag{2.13}$$

*E. Symmetry considerations.* In cases where the basic flow has a velocity distribution which is a symmetric function of $z_0$ (symmetric basic flow) certain simplifications can be made in the problem. In functional notation a symmetric function $F(v)$ satisfies

$$F^s(v) = (F^s(v^*))^*$$

while an antisymmetric function satisfies

$$F^a(v) = -(F^a(v^*))^*.$$

We can combine both relations in a convenient notation

$$F^{sa}(v) = \pm(F^{sa}(v^*))^*$$

where the first superscript corresponds to the upper sign, and the second superscript to the lower sign.

Certain operations performed on a symmetric or antisymmetric function preserve these properties. It can be shown, for example, that the operations of differentiation and integration are symmetry preserving, i.e., the symmetry or antisymmetry of the function remains unchanged. Also, if

$$F^{sa}(v) = \pm(F^{sa}(v^*))^*$$

and we transform to a new variable $u$ such that

$$u(v) = (u(v^*))^*$$

then

$$F^{sa}(u) = \pm(F^{sa}(u^*))^*$$

i.e., the function retains its symmetry or antisymmetry properties in the $u$ plane.

The important consequence of these considerations is embodied in a theorem which will be stated here without proof (for proof see [1, pp. 26 et seq.]).

*Symmetry Theorem*: If the basic flow is symmetric, any perturbation can be represented as the sum of a symmetric and an antisymmetric perturbation each of which satisfies all boundary conditions and so is an admissible perturbation in its own right.

*F. Separation of time dependence.* At this point in the development of the theory, the only restriction placed on the perturbation in the potential is that it shall satisfy all applicable boundary conditions. We shall attack the problem by assuming solutions of the form

$$f_2 = G_1(w_0)e^{\lambda t} + G_2(w_0)e^{\lambda^* t}. \tag{2.14}$$

We anticipate that this choice of time dependence will lead to an eigenvalue problem for the determination of the functions $G_1$ and $G_2$. We expect to find that the boundary conditions can be satisfied only for certain specific values of $\lambda$ and that a general solution will be a sum of all such elementary forms of $f_2$. Our primary concern will be to determine

the magnitude of the real part of all admissible $\lambda$ since this will indicate the stability of the flow. We shall have

| | |
|---|---|
| unstable perturbations for | $\mathrm{Re}\{\lambda\} > 0,$ |
| neutrally stable perturbations for | $\mathrm{Re}\{\lambda\} = 0,$ |
| stable perturbations for | $\mathrm{Re}\{\lambda\} < 0.$ |

It might seem sufficient to assume $f_2$ in the form

$$f_2 = G(w_0)e^{\lambda t} \tag{2.14a}$$

since if $\lambda$ and $\lambda^*$ were both eigenvalues, both would be found among the admissible values of $\lambda$. The form (2.14) has been chosen because it is found that the elementary form (2.14a) is not capable of satisfying all the boundary conditions, whereas the form (2.14) can represent an admissible perturbation.

Substitution of the form for $f_2$ from (2.14) in our previous expression (2.12) for the operator $H$ gives

$$H(w_0) = L_\lambda[G_1]e^{\lambda t} + L_\lambda^*[G_2]e^{\lambda^* t}, \tag{2.15}$$

where

$$L_\lambda[G] = \frac{w_0^2}{\omega}\frac{d^2G}{dw_0^2} + 2\lambda w_0 \frac{dG}{dw_0} + \lambda G(\omega' + \lambda\omega). \tag{2.16}$$

Our free surface boundary condition (2.13) becomes

$$L_\lambda[G_1(w_0)] = (L_{\lambda^*}[G_2(w_0)])^*. \tag{2.17}$$

In the event that the basic flow is symmetric we can decompose $f_2$ into symmetric and antisymmetric components and write

$$f_2 = f_2^s + f_2^a, \tag{2.18}$$

where

$$f_2^{sa} = G_1^{sa}e^{\lambda t} + G_2^{sa}e^{\lambda^* t},$$

and we can show that

$$G_2^{sa}(w_0) = \pm(G_1^{sa}(w_0^*))^*. \tag{2.19}$$

With (2.19) we can eliminate $G_2$ from our free surface boundary condition (2.17) which then becomes

$$L_\lambda[G_1^{sa}(w_0)] = \pm L_\lambda\left[G_1^{sa}\left(\frac{1}{w}\right)\right]. \tag{2.20}$$

This condition is to be applied on the free surface $w_0 w_0^* = 1$ or $w_0^* = 1/w_0$. We can use analytic continuation, however, and demand that it hold over the entire $w_0$ plane.

The remainder of this paper will be devoted to solving the perturbation equation (either (2.17) or (2.20) for several types of problems. The analysis gives rise to relations which are exceptionally long and cumbersome. Due to limitations of space, only the essential features of the solutions will be presented here. For further details the reader may refer to the report mentioned in the footnote to the title. The subscript 0 in $w_0$, which denotes the basic flow, will be dropped in the work that follows.

**3. Jet impinging on a finite plate.** One of the flows which the authors intended to study was that which results when a plate of finite width is placed at right angles to an infinite stream which is uniform at infinity and which has an infinite cavity bounded by constant pressure surfaces downstream of the plate, (the Helmholtz plate problem). Since the velocity is assumed to be the same no matter how one approaches infinity, the points corresponding to upstream and downstream infinity in the physical plane map into one and the same point in the hodograph ($w_0$) plane. In the analysis of the perturbed flow one is then confronted with the problem of applying boundary conditions pertaining to different regions in the physical plane at a single point in the hodograph plane. Mathematically the result of this is that the perturbation quantities exhibit a very irregular behavior at this point; it is found, in fact, that one is led to a differential equation which has an irregular singular point there.

To overcome this difficulty an attempt was made to differentiate between upstream and downstream infinity by artificially separating slightly the source and sink representing these points in the hodograph plane. After applying the boundary conditions the points would be made to coalesce and it was hoped that the results so obtained would be the same as if the original problem had been solved directly.

In the course of this study it was realized that the separation of upstream and downstream infinity could be accomplished in a more direct and rigorous manner by considering the Helmholtz plate as the limiting case of a related and much more general physical problem, namely that of a jet of finite width which impinges on a finite plate and is thereby divided into two jets which diverge from the edges of the plate and tend to become straight jets inclined at $\pm\theta$ to the horizontal as they approach downstream

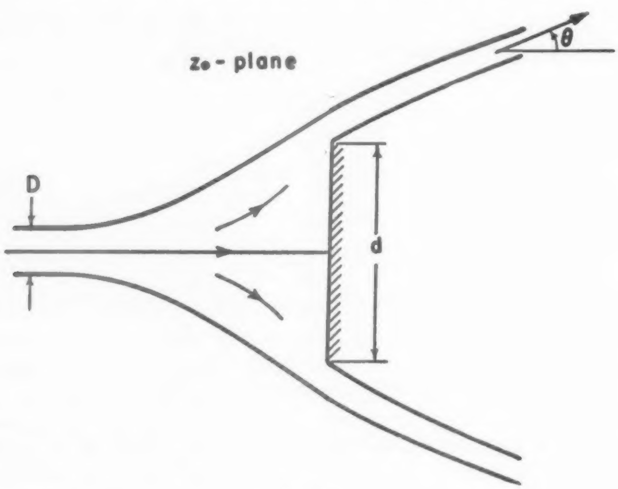


FIG. 1.

infinity (see Fig. 1). The limiting case of infinite jet width then constitutes the proper mathematical formulation of the Helmholtz plate problem.

*A. Basic flow equations.* We begin by writing the potential of the basic flow in the hodograph plane (see Fig. 2). This flow may be thought of as arising from the presence

of unit sources at $w = \pm 1$ and sinks of strength one-half on the unit circle at $w = \pm a$ and $\pm a^*$ where argument of $a = \theta$, the jet inclination at downstream infinity. The potential is then found to be

$$f_0(w) = \log \frac{(w^2 - 1)^2}{(w^2 - a^2)(w - a^{*2})}. \tag{3.1}$$

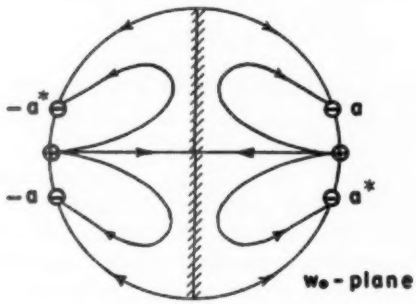


FIG. 2.

Since we are only interested in that part of the hodograph plane which corresponds to the physical flow, it is convenient to introduce a transformation $\zeta = w^2$ such that the right half of the unit circle in the $w$ plane goes into the entire unit circle in the $\zeta$ plane. This transformation has the advantage of reducing the number of singular points on the unit circle.

The potential in the $\zeta$ plane becomes

$$f_0(\zeta) = \log \frac{(\zeta - 1)^2}{(\zeta - b)(\zeta - b^*)}, \tag{3.2}$$

where

$$b = a^2.$$

The asymptotic inclination of the downstream jets, as characterized by a parameter $\beta = \sin \theta$, can be related to the dimensionless ratio $(d/D)$ of plate width to original jet width. This can be accomplished by considering the integral

$$\int_0^{d/2D} dz = \frac{1}{2}\left(\frac{d}{D}\right).$$

In terms of hodograph variables we have

$$dz = \frac{1}{w}\left(\frac{df_0}{dw}\right) dw.$$

Thus

$$\int_0^{d/2D} dz = \int_0^{-i} \frac{1}{w}\left(\frac{df_0}{dw}\right) dw = 1/2(d/D).$$

Now using (3.1) for $f_0(w)$ which involves the unknown sink position $a$, we finally arrive at

$$(d/D) = \pi/2(1 - \sqrt{1 - \beta^2}) + \beta/2 \log \left(\frac{1 + \beta}{1 - \beta}\right). \tag{3.3}$$

The limit cases $\beta = 1$ and $\beta = 0$ correspond to the impinging of a jet of finite width on an infinite plate, and the impinging of an infinite stream upon a finite plate, respectively. These, in turn, correspond to making the sink position $a$ approach $i$ or $+1$, respectively. The steady state form of the potential, (3.1), in these limiting cases, goes over into the known form of the potential with $(d/D) = 0$ or $\infty$ respectively.

*B. Derivation of the perturbation equation.* As discussed in the basic theory we try a solution of the form

$$f_2 = G_1 e^{\lambda t} + G_2 e^{\lambda^* t} \tag{3.4}$$

Our first objective is to narrow the possible choice of functions $G_1$ and $G_2$ by investigating the functional properties which $f_2$ must possess in order to satisfy some of the conditions of the problem.

(1) Wall streamline condition. The boundary condition on the imaginary axis (map of the plate) in the $w$ plane is that the perturbed flow have no component normal to the axis. It is convenient to transform this condition to the real axis. To do this we rotate the hodograph plane by putting $\eta = iw$. The boundary condition will now be satisfied if $w_2/w$ is real on $\eta$ real. Denoting differentiation with respect to $\eta$ by the subscript $\eta$ and evaluating $w_2/w$ we have that

$$\frac{\eta(\eta^2 + 1)(\eta^2 + b)(\eta^2 + b^*)}{2\eta^2\{2(\eta^2 + b)(\eta^2 + b^*) - (\eta^2 + 1)[(\eta^2 + b) + (\eta^2 + b^*)]} f_{2\eta}$$

must be real on $\eta$ real. Since the coefficient of $f_{2\eta}$ in the expression above is itself real for real $\eta$, we infer that $f_{2\eta}$ must be real on $\eta$ real and hence a symmetric function of $\eta$ in $|\eta| < 1$.

(2) Analyticity and symmetry considerations. Since the basic flow is everywhere regular in $|\eta| < 1$, $f_{2\eta}$ must also be regular in $|\eta| < 1$. Upon integrating $f_{2\eta}$ we then find $f_2(\eta)$ is both regular and symmetric in $|\eta| < 1$, i.e.,

$$f_2(\eta) = (f_2(\eta^*))^*. \tag{3.5}$$

Since the basic flow is symmetric we may decompose the perturbations into symmetric and antisymmetric components, i.e.,

$$f_2 = f_2^s + f_2^a \qquad \text{where} \qquad f_2^{sa}(\zeta) = \pm(f_2^{sa}(\zeta^*))^*.$$

These relations hold in the $\zeta$ and $w$ planes since the transformation from $w$ to $\zeta$ preserves symmetry.

In the $\eta$ plane ($\eta = i\zeta^{1/2}$) the above becomes

$$f_2^{sa}(\eta) = \pm(f_2^{sa}(-\eta^*))^*,$$

where the superscripts still refer to the symmetric and antisymmetric components of $f_2$ referred to the $\zeta$ (or $w$) plane. Using the relation (3.5) we have

$$f_2^{sa}(\eta) = \pm f_2^{sa}(-\eta).$$

Thus, a $\zeta$-plane symmetric or antisymmetric perturbation is represented by an even or odd function of $\eta$, respectively, and we may write

$$f^s = \sum_0^\infty a_k \eta^{2k}, \qquad f^a = \sum_0^\infty b_k \eta^{2k+1}.$$

Since $\zeta = -\eta^2$ these relations transform in the $\zeta$ plane to

$$f_2^s = F^s(\zeta), \qquad f_2^a = \zeta^{1/2}F^a(\zeta).$$

where $F^s$ and $F^a$ are regular functions of $\zeta$ in $|\zeta| < 1$. We can also write

$$f_2^{sa} = G_1^{sa}e^{\lambda t} + G_2^{sa}e^{\lambda^* t}$$

where we know that, by virtue of the symmetric basic flow,

$$G_2^{sa}(w) = \pm(G_1^{sa}(w^*))^*.$$

Since $\zeta = w^2$ is a symmetric function of $w$, the above symmetry relation also holds in the $\zeta$ plane (see 2E), i.e.,

$$G_2^{sa}(\zeta) = \pm(G_1^{sa}(\zeta^*))^*.$$

The four functions $G_1^{sa}$ and $G_2^{sa}$ can be replaced by two functions $g^{sa}$ by the following relations which are found to be consistent with the above considerations.

$$\begin{aligned} G_1^s &= g^s(\zeta), & G_2^s &= (g^s(\zeta^*))^* \\ G_1^a &= g^a(\zeta), & G_2^a &= -(g^a(\zeta^*))^* \end{aligned} \tag{3.6}$$

where $g^s$ and $\zeta^{-\frac{1}{2}}g^a$ are regular functions of $\zeta$ in $|\zeta| < 1$.

(3) The free surface boundary condition. For a symmetric basic flow the free surface condition in the $\zeta$ plane is

$$L_\lambda[G_1^{sa}(\zeta)] = L_{\lambda^*}\left[G_1^{sa}\left(\frac{1}{\zeta}\right)\right], \tag{3.7}$$

where the form of the differential operator $L_\lambda$ in the $\zeta$ plane is found from (2.16) and (3.1). It is not reproduced here because of space limitations.

If we substitute the forms (3.6) for $G_1^{sa}$ in (3.7), we arrive at the following relations

$$h^s(\zeta) = h^s\left(\frac{1}{\zeta}\right) \tag{3.8}$$

$$h^a(\zeta) = -h^a\left(\frac{1}{\zeta}\right) \tag{3.9}$$

where

$$h^{sa}(\zeta) = L_\lambda[g^{sa}(\zeta)] \tag{3.10}$$

The form of the differential operator $L_\lambda$ and our knowledge of the functional behavior of $g^s$ and $g^a$ in $|\zeta| < 1$ together with (3.10) show that, in $|\zeta| < 1$, $h^s(\zeta)$ is a regular function of $\zeta$ and that $h^a(\zeta)$ behaves like $\zeta^{\frac{1}{2}}$ multiplied by a function with a simple pole at $\zeta = 0$. For convenience we shall represent $h^s$ and $h^a$ in the following manner

$$\begin{aligned} h^a(\zeta) &= \frac{\zeta - 1}{\zeta^{1/2}} \sum_{-\infty}^{\infty} a_k\left[\frac{(\zeta-1)^2}{(\zeta-b)(\zeta-b^*)}\right]^k, \\ h^s(\zeta) &= \sum_{-\infty}^{\infty} b_k\left[\frac{(\zeta-1)^2}{(\zeta-b)(\zeta-b^*)}\right]^k. \end{aligned} \tag{3.11}$$

with $a_k$ and $b_k$ unknown constants. These forms for $h^s$ and $h^a$ possess the required behavior in $|\zeta| < 1$, and in addition satisfy the functional relations (3.7) termwise.

Our aim is to find functions $g^{sa}(\zeta)$ satisfying the relation (3.10) with $h^{sa}(\zeta)$ having the forms given in (3.11). To do this we regard (3.10) as an inhomogeneous differential equation for $g^{sa}(\zeta)$. We note that the inhomogeneous term $h^{sa}(\zeta)$ is known in form only and hence the solutions $g^{sa}(\zeta)$ will retain some arbitrariness which, for a given problem, ought to be determined by the initial conditions.

*C. Solutions of the perturbation equation.* Equation (3.10) is a second order linear differential equation of the Fuchsian type [7] with regular singular points at $\zeta = 0, 1, b, b^*$ and $\infty$. We can first find the complementary solution about the origin, which we can denote as $AK^{(1)} + BK^{(2)}$, by the standard methods of solution in series. Knowing the complementary solutions, the particular integral is readily found by the method of variation of parameters, viz.

$$\text{P.I.} = K^{(1)} \int_{\zeta_0}^{\zeta} \frac{p(\zeta)K^{(2)}\,d\zeta}{W(K^{(1)}, K^{(2)})} - K^{(2)} \int_{\zeta_0}^{\zeta} \frac{p(\zeta)K^{(1)}(\zeta)\,d\zeta}{W(K^{(1)}, K^{(2)})}, \tag{3.12}$$

where

$$p(\zeta) = \left[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-b^*)}\right]h^{sa}(\zeta),$$

and $W(K^{(1)}, K^{(2)})$ is the Wronskian of the two solutions which can be found from the differential equation and is $\Lambda\zeta^{-\frac{1}{2}}(\zeta-1)^{-4\lambda}(\zeta-b)^{2\lambda}(\zeta-b^*)^{2\lambda}$ where $\Lambda$ is a known constant. The point $\zeta_0$ is an arbitrary ordinary point of the differential equation which we shall choose as $\zeta_0 = -1$. This solution will be valid up to the nearest singular point, i.e., within the unit circle. Since boundary conditions will have to be applied not only within, but also on the unit circle, in particular at the singular points, we must find solutions valid at these points. To do this we study the indicial equation and hence the form of the complementary solution appropriate to the point in question and then match this form of the solution with the solution around the origin by the process of analytic continuation. A new form of the particular integral may then be found. This method must be continued until we obtain solutions near all points at which boundary conditions must be applied.

To conserve space, we shall write the form of the solution around the origin only. It is

$$g^{sa}_R(\zeta) = A^{sa}_R K^{(1)} + B^{sa}_R K^{(2)} + \sum_{r=0}^{\infty} E^{sa}_r K^{sa}_{R+r}\,, \tag{3.13}$$

where $A^{sa}_R$, $B^{sa}_R$ and $E^{sa}_r$ are unknown constants. The term $E^s_R\, K^s_{R+r}$ is a particular integral when $h^s(\zeta)$ in (3.12) is replaced by, (see (3.11)),

$$h^s_{R+r} = b_{R+r}\left[\frac{(\zeta-1)^2}{(\zeta-b)(\zeta-b^*)}\right]^{R+r}.$$

Similarly $E^a_r K^a_{R+r}$ is a particular integral when $h^a(\zeta)$ in (3.12) is replaced by

$$h^a_{R+r} = \frac{\zeta-1}{\zeta^{1/2}}\, a_{R+r}\left[\frac{(\zeta-1)^2}{(\zeta-b)(\zeta-b^*)}\right]^{R+r}.$$

The index $R$, which is an integer used to designate the solutions, is chosen such that $E^{sa}_0$ is the first non-zero $E^{sa}$.

*D. Application of the boundary conditions.* At this stage the symmetric and antisymmetric perturbations found as solutions of the perturbation equation satisfy the wall streamline and free surface boundary conditions. We shall now investigate the restrictions to be placed on these solutions in order that they may satisfy all remaining boundary conditions. We shall first discuss the antisymmetric solutions.

(1) Anti-symmetric solutions. (*a*) The edges of the plate ($\zeta = -1$).

We demand that the inner surfaces of the jets diverging from the edges of the plate continue to originate there in the perturbed state. Since $z_1$ is the perturbation in the space coordinate which takes the basic flow free surface into the perturbed free surface (see Eq. (2.10) in 2D), this condition will be satisfied if

$$z_1 = \frac{1}{w'} \{i\chi - f_2^{\cdot} - f_2'\} = 0 \qquad \text{at} \qquad \zeta = -1.$$

Using (3.4) and (3.6) we find this implies $g_I^a(-1) = 0$. The solution $g_R^a$ valid near $\zeta = -1$ will satisfy this if an equation of the form

$$C_1 A_R^a + C_2 B_R^a = 0 \tag{3.14}$$

is satisfied with $C_1$ and $C_2$ non-zero constants.

(*b*) At the point $\zeta = 0$. We have all ready specified the behavior of $g^a(\zeta)$ in $|\zeta| < 1$. We can insure that the solution $g_R^a$ possess this behavior if $\zeta^{-\frac{1}{2}} g^a$ is a regular function of $\zeta$ as $\zeta \to 0$.

This boundary condition can be satisfied by equating to zero the terms in the solution $g_R^a$ valid near $\zeta = 0$ which do not have the required $\frac{1}{2}$ order behavior. This results in two further linear homogeneous equations to be satisfied by the unknown constants which are of the form

$$B_R^a + \sum_{r=0}^{\infty} \alpha_r E_r^a = 0 \tag{3.15}$$

and

$$\sum_{r=0}^{\infty} \delta_r E_r^a = 0, \tag{3.16}$$

where $\alpha_r$ and $\delta_r$ are non-zero constants.

(*c*) Downstream infinity ($\zeta = b$ and $\zeta = b^*$). In the basic flow the jets diverging from the edges of the plate asymptotically become straight uniform jets. One can investigate the behavior of a straight jet when subjected to small disturbances using the approach described in Lamb's Hydrodynamics (Chapter IX). Such a flow is found to be neutrally stable and any disturbance is propagated downstream unchanged with the velocity of the jet. We shall demand that our perturbed flow behave like a straight jet as we approach downstream infinity. Hence, an observer moving with the asymptotic jet velocity will see no change in the velocity perturbation $w_2$. If we express this fact by means of the material derivative of $w_2$ we find that the boundary condition demands that $\lim_{\zeta \to b}(\zeta - b)^{-\lambda} g^a$ and $\lim_{\zeta \to b^*}(\zeta - b^*)^{-\lambda^*} g^a$ exist.

An examination of the solutions valid about $b$ and $b^*$ shows them to be derivable from each other by a simple interchange of the roles of $b$ and $b^*$ and hence only one of the conditions has to be applied.

It is found that the terms of the complementary solution satisfy the boundary con-

dition for any value of $\lambda$. By investigating the behavior of the integrals making up the particular integral, it can be shown that the term in $(\zeta - b)^{-\lambda} g_R^a$ with the least real part of the exponent of $(\zeta - b)$ has a non-zero coefficient and hence the boundary condition demands that the real part of the exponent be greater than zero. This implies that

$$R + \operatorname{Re}\{\lambda\} \leq 1 - r_{\max} \tag{3.17}$$

where $r_{\max}$ is the greatest $r$ for which we must have $E_r^a \neq 0$. From (3.16) we see that, since $E_0^a \neq 0$, $r_{\max}$ must be at least as great as one. Hence, the inequality becomes

$$R + \operatorname{Re}\{\lambda\} \leq 0 \tag{3.18}$$

(*d*) Upstream infinity ($\zeta = 1$). The basic flow originates from a source point at upstream infinity. Since we do not wish the perturbations to alter the fundamental nature of the entire flow we shall demand that the perturbation velocity $w_2$ and the perturbation pressure $p_2$ vanish at upstream infinity. These conditions will be satisfied if $(\zeta - 1)g_\zeta^a$ and $g^a$, respectively, vanish as $\zeta \to 1$.

Since the point $\zeta = 1$ is a regular singular point of the differential equation, however, our knowledge of the behavior of the solution at such a point shows that these conditions are redundant and only one, say the latter, will be applied.

We must show that in the neighborhood of $\zeta = 1$ the expression for $g_R^a$ consists of terms which go to zero as $\zeta \to 1$ or that any terms which do not go to zero have a coefficient which is zero. If one examines the solution $g_R^a$ near $\zeta = 1$, one finds that the dominant term from the complementary solution behaves like $(\zeta - 1)^{-2\lambda}$ while the dominant term contributed by the particular integral is either $(\zeta - 1)^{-2\lambda}$ or $(\zeta - 1)^{2R+1}$, depending upon the relative magnitude of $R$ and $\operatorname{Re}\{\lambda\}$. As yet, $R$ and $\operatorname{Re}\{\lambda\}$ are restricted only by the inequality (3.18). We can now proceed to investigate the various manners in which the boundary condition at $\zeta = 1$ may be satisfied depending upon which of the following inequalities is applicable:

$$\begin{aligned} &(i) \quad 2R + 1 < \operatorname{Re}\{-2\lambda\}, \\ &(ii) \quad 2R + 1 = \operatorname{Re}\{-2\lambda\}, \\ &(iii) \quad 2R + 1 > \operatorname{Re}\{-2\lambda\}. \end{aligned} \tag{3.19}$$

Consider, for example, Case (*i*). The term $(\zeta - 1)^{2R+1}$ has a non-zero coefficient; hence, the boundary condition demands $2R + 1 > 0$ which gives $R = 0, 1, 2, \cdots$ and then the inequality shows $\operatorname{Re}\{\lambda\} < -\frac{1}{2}$. In a similar manner one finds in every case that $\operatorname{Re}\{\lambda\} < 0$.

(2) Symmetric solutions. The application of the boundary conditions to the symmetric perturbations proceeds in the same manner as in the antisymmetric ones. We again find no admissible perturbations with $\operatorname{Re}\{\lambda\} > 0$.

*E. Admissible complementary solutions.* One may consider the possibility of satisfying the boundary conditions with solutions containing only terms of the complementary solution. In both the antisymmetric and symmetric case one can readily show, however, that there are no such solutions capable of satisfying all the boundary conditions.

*F. Conclusions as to the stability of the basic flow.* Although we have not found a one-dimensional continuum of eigenvalues $\lambda$, our present analysis has been able to restrict

the possible values of $\lambda$ to the left half of the complex $\lambda$ plane. This shows that any admissible perturbation has Re $\{\lambda\} \leq 0$ and we conclude that a jet impinging normally upon a finite plate gives a neutrally stable or stable flow configuration.

The results of this analysis are valid for any finite but non-zero ratio of plate width to jet width $(d/D)$. By making this ratio as small as we please, we can consider the flow of a jet of arbitrarily great width past a finite plate. But this is a proper physical interpretation of the Helmholtz plate problem since one never has a truly infinite stream in reality. From this point of view, the stability conclusions reached in this section apply to the Helmholtz plate problem.

On the other hand, we find a different situation if we make the ratio $(d/D)$ very large but still finite. In this case the point $\zeta = -1$ continues to be an ordinary point of the perturbation differential equation and the boundary condition $z_1 = 0$ must always be satisfied there. However, if one actually sets $d/D = \infty$, the basic flow of this section becomes that of a jet impinging on an infinite wall. Now the point $\zeta = -1$ is a regular singular point corresponding to downstream infinity and the boundary condition $z_1 = 0$ no longer applies. A reexamination of the work of the present section will show that it was the presence of this additional boundary condition which precluded the existence of any non-trivial unstable perturbations. Thus, as far as stability is concerned, the limiting case $(d/D) = \infty$ corresponding to the impinging of a finite jet on a plate of extremely large width does not correspond to the case of a jet impinging on a truly infinite wall. The latter problem is equivalent to that of perturbations which are symmetric about the vertical axis of two equal and opposite jets impinging upon each other. It is considered in Section 4.

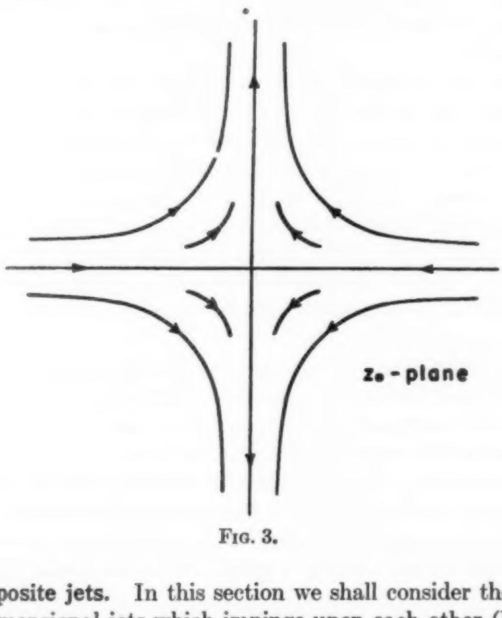


FIG. 3.

**4. Equal and opposite jets.** In this section we shall consider the flow of two equal and opposite two-dimensional jets which impinge upon each other (Fig. 3).

The potential of the basic flow in the hodograph plane is readily found as that due

to two sources at $w = \pm 1$ and two sinks at $w = \pm i$, all of the same strength (Fig. 4).

The flow possesses a vertical axis of symmetry as well as a horizontal one and the perturbation problem is conveniently divided into two problems, that of perturbations symmetric and antisymmetric about the vertical axis of symmetry. These correspond in term to demanding that $w_2/w$ be either real or purely imaginary, respectively, on the imaginary axis in the hodograph plane.

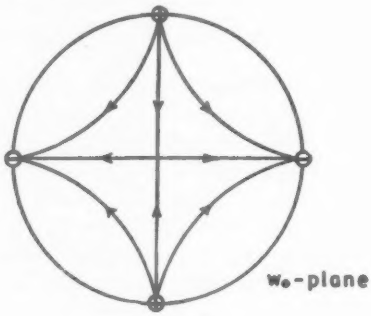


FIG. 4.

Now if one proceeds as in Section 3 to consider the boundary conditions on $w_2/w$ mentioned above along with analyticity and symmetry requirements and the free surface condition, one is again led to perturbations which are solutions of an inhomogeneous differential equation. In terms of a new variable $\zeta = w^2$ the equation can be reduced to a standard form known as Heun's equation [10] and [11] possessing regular singular points at $\zeta = 0, 1, -1$, and $\infty$.

Again proceeding as in Section 3 we can apply the remaining physical boundary conditions to the solutions of the perturbation equation. It is found that for perturbations which are symmetric about the vertical axis of symmetry, there exist non-trivial solutions with $0 \leq \text{Re}\,\{\lambda\} < 1$. All other admissible perturbations have $\text{Re}\,\{\lambda\} < 0$.

**5. Generalized orifices.** Each of the flows to be considered in this section represents the draining of an infinite reservoir through an orifice (Fig. 5). The sides of the orifice are made up of two semi-infinite planes inclined to each other at an angle of $2\,\pi/n$ radians, where $n = 2^p$, $p = 0, 1, 2, \cdots$. When $p = 0$ the configuration becomes the Borda mouthpiece which has been treated by Ablow and Hayes, and hence this section is essentially a generalization of that problem.

The similarity of these flows for various $n$ is seen by examining the potential of the basic flow in the hodograph plane. The flow in the physical plane is mapped into a sector of the unit circle in the hodograph plane bounded by radii inclined at an angle of $\pm\,\pi/n$ radians to the positive real axis (Fig. 6).

To find the potential for which the circular arc and radii are streamlines, we image the sector to cover the entire unit circle. The basic flow potentials for all $n$ are similar in that they may be thought of as arising from the presence of a source at the origin and sinks on the unit circle at the $n$ roots of unity. Thus, the potential may be written as

$$f_0(w) = \log \frac{w^{n/2}}{(w^n - 1)^2}. \tag{5.1}$$

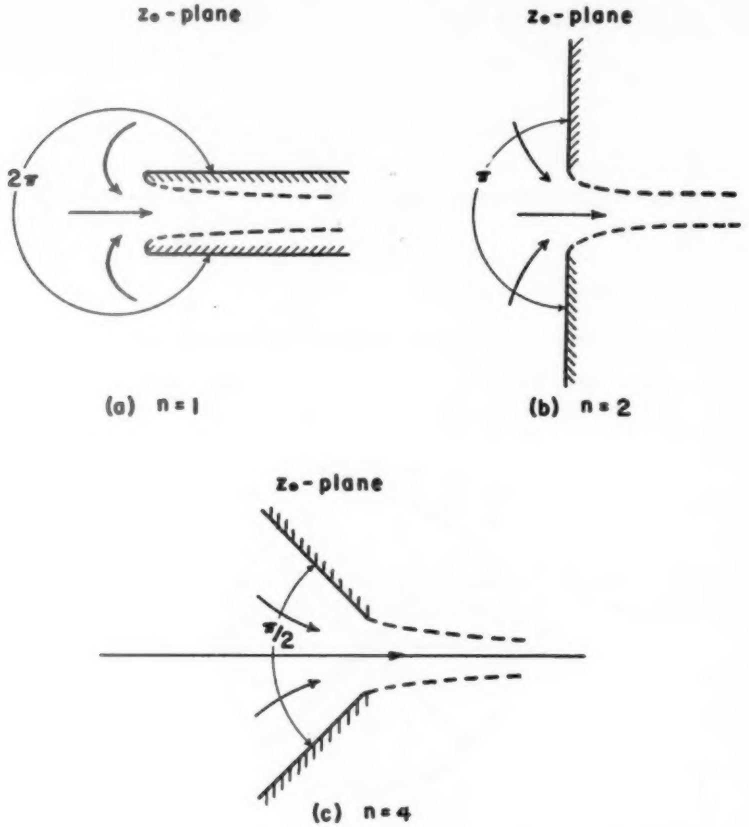


FIG. 5.

But if we transform to a new variable $\zeta = w^n$ the potential becomes

$$f_0(\zeta) = \frac{1}{2} \log \frac{\zeta}{(\zeta - 1)^2} \tag{5.2}$$

which provides a single representation for the potential of all the flows considered here. We are now in a position to investigage the stability of the flows for all $n$ by a single analysis.

In this problem the wall streamline and free surface conditions lead to an inhomogeneous second order differential equation governing the perturbations which can be put into the standard form of the hypergeometric equation having regular singular points at $\zeta = 0$, 1 and $\infty$.

After writing the solutions to the perturbation equation, the application of the boundary conditions, which are similar to those applied in Section 3, shows that all admissible perturbations have Re $\{\lambda\} < 0$. One difference noted here is that it appears possible to satisfy all boundary conditions with an antisymmetric perturbation made

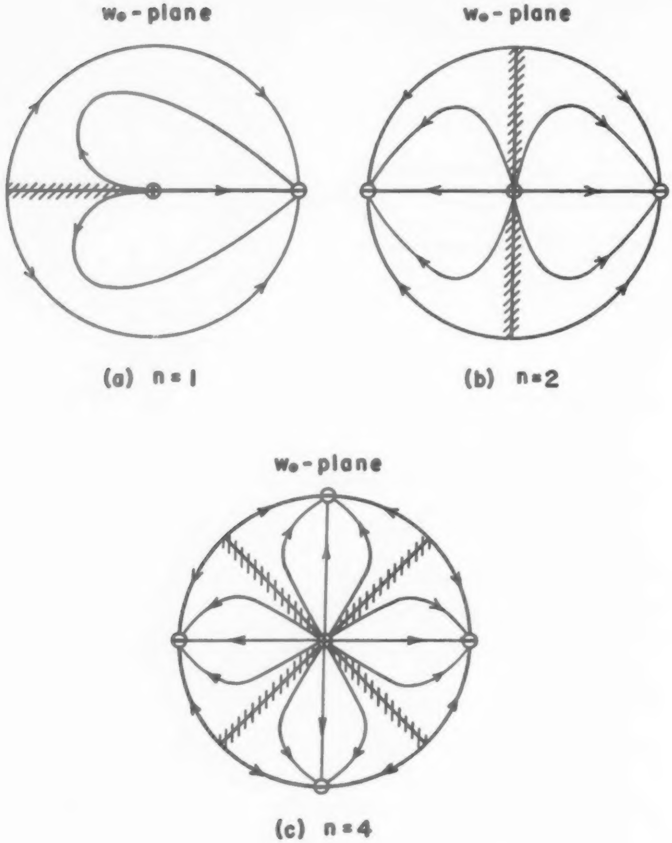


FIG. 6.

up only of terms from the complementary solutions. This is possible if $\lambda$ takes on the values

$$\lambda = -(2N + 1) \pm \left[\frac{2}{n}(2N + 1)\right]^{1/2}, \qquad \begin{array}{l} N = 0, 1, 2, \cdots \\ n = 1, 2, 4, 8, \cdots \end{array}$$

We note that for $N = 0$ and $n = 1$, $\lambda = -1 + 2^{1/2}$ which has a positive real part.

Thus, for all $n$ but $n = 1$ we have only stable perturbations, while for $n = 1$, corresponding to the Borda mouthpiece flow, there exists an isolated unstable perturbation.

**6. Hollow vertex bounded by cylindrical walls.** This problem is a generalization of a problem treated by Ablow and Hayes who considered the unbounded flow about a hollow vortex.

The basic flow is a cyclic irrotational motion with circular streamlines bounded on the outside by a solid circular wall and on the inside by a concentric circular hollow

vortex forming a constant pressure surface. The basic flow potential in the hodograph plane is simply

$$f_0(w) = i \log w. \tag{6.1}$$

This flow has an antisymmetric velocity distribution and hence we may no longer decompose an arbitrary perturbation into symmetric and antisymmetric components which satisfy the boundary conditions separately.

Due to the simplicity of the basic flow potential, however, one can assume the form of the perturbation potential as

$$f_2 = G_1(w)e^{\lambda t} + G_2(w)e^{\lambda^* t}, \tag{6.2}$$

where the forms for $G_1$ and $G_2$ may be deduced from a knowledge of the basic flow behavior as

$$G_1(w) = B_1 \log (w) + \sum_{-\infty}^{\infty} a_r w^r,$$

$$G_2(w) = B_2 \log (w) + \sum_{-\infty}^{\infty} b_r w^r,$$

with $B_1$, $B_2$, $a_r$ and $b_r$ arbitrary constants.

Applying the boundary conditions that the circular wall remains a streamline and that the perturbation satisfies the free surface condition (2.17) we find an infinite set of eigenvalues

$$\lambda_n = -i(n \pm [N(n)]^{1/2})$$

for all $n$, where

$$N(n) = n\left[\frac{1 - \alpha^{2n}}{1 + \alpha^{2n}}\right]$$

and $\alpha$ is the ratio of the radius of the hollow vortex to that of the wall; $\lambda_n$ is purely imaginary and we conclude that the basic flow is a neutrally stable configuration.

Corresponding to each value of $n$ we can now find an elementary perturbation $f_{2n}$ and can generate any admissible perturbation by summing the elementary solutions for all $n$. With $f_2$ known we can evaluate the perturbation of the free surface from (2.10). Aside from the arbitrariness introduced by the function $\chi$ the perturbation is made up of a wave pattern whose components travel at angular velocities equal to $1 \pm [N(n)]^{1/2}/n$, i.e., they either lead or lag the basic flow by an angular velocity of $[N(n)]^{\frac{1}{2}}/n$.

An interesting analogy with the propagation of gravity waves in water is found when the ratio of the depth of fluid between the vortex surface and the walls, to the radius of the vortex surface is made very small. At the same time we consider a fixed but arbitrary ratio of depth of fluid to wave length $\gamma$ of the disturbance. The propagation velocity of the disturbance is found to be approximately

$$c^2 = \frac{\gamma}{2\pi a} \tanh \frac{2\pi h}{\gamma}$$

where $h$ is the depth of fluid between the free surface and the wall. This is identical with the velocity of the propagation of a gravity wave in water if $(1/a)$ is replaced by the

acceleration of gravity $g$. The term $(1/a)$ represents approximately the centrifugal acceleration throughout the fluid for the case of small depth.

This problem has also been solved by Lord Kelvin in a different manner and after a change of notation the results presented here are seen to agree in detail with those found by Kelvin.

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# CLOSURE WAVES IN HELICAL COMPRESSION SPRINGS WITH INELASTIC COIL IMPACT[1]

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**Summary.** This paper deals with the problem of spring surges taking into account coil closure. This may occur in many cases of compression springs subject to impact. Inelastic coil on coil impact conditions are assumed. The simple theory of spring surges is adopted wherein only the motion of the spring wire parallel to the axis of the spring is considered and the assumption made that each element of the spring satisfies the force-longitudinal strain relation of the whole spring, before coil closure occurs. The basic theory of coil closure with inelastic impact conditions has been given by Lee [1][3]. The spring is initially at rest and unstrained with one end fixed and the other (the impact end) is given an impulsive velocity and then either maintained at this velocity or decelerated at a constant rate. The case of a mass attached to the impact end is also considered. Conditions are obtained for which partial or complete closure of the spring occurs.

**1. Introduction.** We are concerned with the motion of a helical compression spring as depicted diagramatically in Fig. 1, the end $x = l$ being fixed. Following [1] we use as

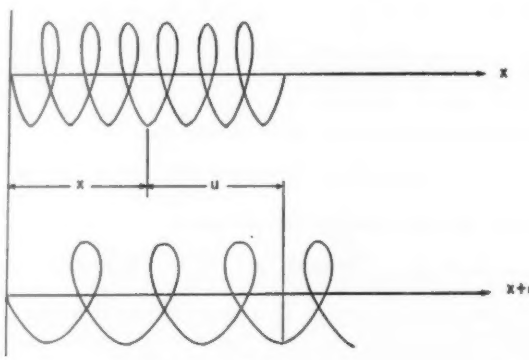


FIG. 1.

an independent space variable the position $x$ along the unstrained spring. The displacement of an element of the spring in position $x$ is taken as $u(x, t)$, so that the position of the element at time $t$ is given by $(x + u)$. If $f$ is the compressive force transmitted

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[3]Numbers in square brackets refer to the Bibliography at the end of the paper.

across the section $x$, then the equation of motion of an element $dx$ of the spring is

$$u_{tt} = -\frac{1}{m} f_x , \tag{1}$$

where $m$ is the mass of the spring per unit original length. This theory is analogous with that of longitudinal waves in a linear elastic material and here $f$ replaces the nominal stress and $m$ the density. The nominal compressive strain of an element of the spring is $\epsilon = -u_x$. Before closure occurs there is a linear relation between the stress and the strain given by

$$f = E\epsilon, \tag{2}$$

where $E$ is a constant depending on the dimensions and material of the spring.

Substituting in (1) gives the linear wave equation for $u$,

$$u_{tt} - c_0^2 u_{xx} = 0, \tag{3}$$

where $c_0 = \sqrt{E/m}$ is the constant velocity of wave propagation.

The velocity $v$ of an element of the spring is given by $v = u_t$. The characteristics in the $(x,t)$-plane are the straight lines $(x \pm c_0 t) = $ constant. The corresponding characteristic relations are $c_0\epsilon \mp v = $ constant, or $mc_0 v \mp f = $ constant.

Now let us introduce the dimensionless variables

$$T = \frac{c_0}{l} t; \qquad X = \frac{x}{l}; \qquad V = \frac{v}{v_0}; \qquad \Sigma = \frac{f}{f_0}; \qquad \Phi = \frac{\epsilon}{\epsilon_0},$$

where $l$ is the length of the unstrained spring, $v_0$ is the initial impulsive velocity of the impact end, $f_0 = mc_0v_0$ and $\epsilon_0 = v_0/c_0$.

Then the characteristic lines in the $(X,T)$-plane are

$$X \pm T = \text{constant}, \tag{4}$$

and the corresponding characteristics relations are

$$V \mp \Sigma = \text{constant, or } V \mp \Phi = \text{constant}. \tag{5}$$

We now consider closure of the coils. This takes place when the approach of sections of adjacent coils with the same angular position is equal to the initial separation of the coils. This condition is expressed analytically in the form

$$u(x) - u(x + p) = p - d, \tag{6}$$

where $p$ is the pitch of the coils and $d$ is their thickness.

Since we are considering in this paper only inelastic coil impact conditions so that the coils remain in contact after closure (until separated by the elasticity of the spring) and we are concerned only with the average effects around a single coil it is permissible to replace the finite difference condition in (6) by a derivative (see [1] for a fuller discussion) so that coil closure is to be associated with a fixed upper bound for the compressive strain $\epsilon$, giving the condition

$$\epsilon = -u_x \leq (p - d)/p = \epsilon_{\max} . \tag{7}$$

or

$$\Phi \le K \tag{8}$$

where $K = \epsilon_{max}/\epsilon_0$ . Hence we have

$$\begin{aligned} \Phi &= \Sigma, \qquad \text{for} \qquad \Sigma \le K \\ \Phi &= K, \qquad \text{for} \qquad \Sigma > K \end{aligned} \tag{9}$$

since an increase in the compressive force may take place when coil closure has occurred but no increase can take place in $\Phi$ according to (8).

The relation between $\Sigma$ and $\Phi$ is shown in Fig. 2. Considerations of continuity and

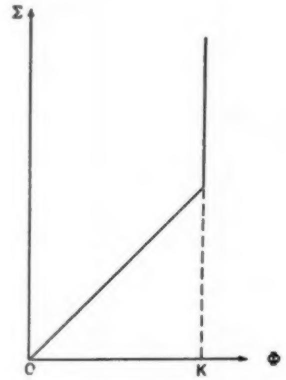


Fig. 2.

momentum across wave fronts of closure yield the conditions

$$\left.\begin{aligned} |\Delta v| &= c\cdot\Delta\epsilon \\ \Delta f &= mc\cdot|\Delta v| \end{aligned}\right\} \tag{10}$$

where $\Delta$ represents the discontinuity of the variable due to the passage of the wave, and $c$ is the magnitude of the wave velocity, relative to the unstrained spring.

In terms of dimensionless variables Eq. (10) becomes

$$\begin{aligned} |\Delta V| &= \Omega\cdot\Delta\Phi \\ \Delta\Sigma &= \Omega\cdot|\Delta V| \end{aligned} \tag{11}$$

where $\Omega = c/c_0$ . Equations (4), (5), (9) and (11) are used in determining the solution of the problem for prescribed boundary conditions.

**2. Impulsive velocity at impact end maintained.** In this section we consider the case in which the end $x = 0$ is given an impulsive velocity $v_0$ at time $t = 0$ and this velocity is maintained, until complete closure of the spring has taken place. The spring is taken to be initially unstrained and at rest. Until closure occurs the configuration in

the $(X,T)$-plane is as shown in Fig. 3, $X = 0$ corresponding to the impact end and $X = 1$ to the fixed end of the spring. Throughout the motion $W = V|_{x=0} = 1$.

A wave of velocity, stress and strain discontinuity is propagated along the spring with constant speed $c_0$, commencing at the impact end at time $t = 0$ and being reflected alternatively at the ends. Using Eqs. (4) and (5) we find that

$$V = \tfrac{1}{2}\{1 - (-1)^r\}, \qquad \Sigma = r \qquad (r \geq 0),$$

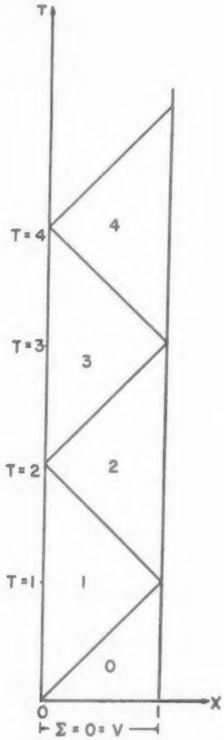


FIG. 3.

in region $r$ of the $(X,T)$-plane. It is seen that after the passage of the discontinuity wave front the stress at a certain position in the spring is increased by an amount $f_0$ and remains unchanged until the wave again passes. The corresponding velocity is alternatively $v_0$ and 0.

Now closure does not occur so long as $\Sigma < K$. Since $\Sigma = r$ in region $r$ it follows that closure must eventually occur, as is otherwise evident. Further, when closure commences it does so from an end of the spring.

Let $n < K \leq n + 1$ ($n \geq 0$, integer). Then closure commences at $T = n$ and from the impact end if $n$ is even, from the fixed end if $n$ is odd. We first consider the case when $n$ is even. Since we are assuming inelastic coil impact conditions the coils do not separate after closure so that at $T = n + \tau$ a portion of the spring $0 \leq X \leq \chi$ is moving as a

rigid body with uniform speed equal to that of the impact end. It follows that the stress throughout this portion is constant.

Now the coils in front of the closure wave are at rest and the stress is given by $\Sigma = n$. Hence in (11) we have $| \Delta V | = 1$, and $\Delta\Phi = (K - n)$, using (8) and (9). Also $\Omega = d\chi/d\tau$. So from (11) we obtain

$$\Omega = 1/(K - n), \qquad \Delta\Sigma = 1/(K - n).$$

Hence the closure wave velocity is constant as shown in Fig. 4(a) (for the case $n = 2$).

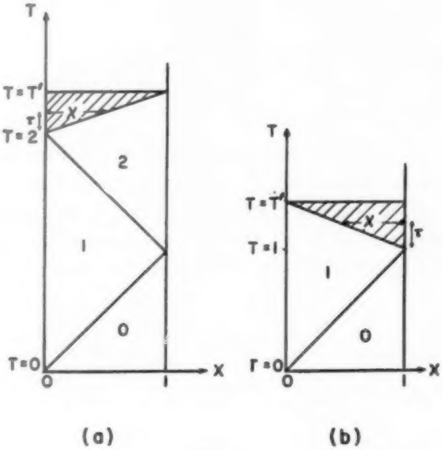


FIG. 4.

The shaded portion depicts coil closure. Complete closure of the spring has taken place at $\tau = (K - n)$, and then $T = T' = K$.

The stress at $X = \chi$ is given by $n + \Delta\Sigma = \{n + 1/(K - n)\}$. If $K$ is close to $n$ this stress is large, but so also is $\Omega$. (Note $\Omega \geq 1$). We have in fact,

$$\int_{T=n}^{T=K} \Delta f \, dt = f_0 \frac{l}{c_0} \int_{T=n}^{K} \Delta\Sigma \, dT = \frac{lf_0}{c_0} = mlv_0$$

$$= \text{(weight of spring)} \times \text{(impact velocity)}.$$

When $n$ is odd closure commences from the fixed end. Ahead of the closure wave the stress is given by $\Sigma = n$ and the velocity is $v_0$, so $V = 1$. In this case the closed portion is at rest, so again we have $| \Delta V | = 1$ and $\Delta\Phi = (K - n)$. Hence the closure wave velocity is given by $\Omega = 1/(K - n)$, as before, and the stress in the closed portion is again $\{n + 1/(K - n)\}$.

The $(X,T)$-plane is shown in Fig. 4(b) for the case $n = 1$.

**3. Impact with constant retardation.** In this section we consider the case in which the end $x = 0$ is retarded at a uniform rate $a$, after being given an impulsive velocity $v_0$ at time $t = 0$. The spring is again taken to be initially unstrained and at rest. Until closure occurs the configuration in the $(X,T)$-plane is as shown in Fig. 3. During the motion the velocity at the impact end is given by $W = (1 - \lambda T)$, where $\lambda = al/c_0v_0 > 0$.

As in the previous case a wave of velocity, stress and strain discontinuity is propagated along the spring with constant speed $c_0$, commencing at time $t = 0$ from the impact end and being reflected alternately at the ends. By making use of Eqs. (4) and (5) we find expressions for $V$ and $\Sigma$ as given in the Appendix. We find that, if closure does occur, then it takes place from an end of the spring and moreover from that end at which reflection of the wave of discontinuity has just occurred.

We next investigate the time at which closure occurs, if at all. Let $\Sigma_r^*$ denote the value of $\Sigma$ at $T = (r - 1) + 0$, $(r \geq 1)$, at the end $X = 0$ if $r$ is odd and at the end $X = 1$ if $r$ is even.

Then

$$\left.\begin{aligned} \Sigma_{2k+1}^* &= \{(2k + 1) - 2k^2\lambda\} \\ \Sigma_{2k+2}^* &= 2(k + 1)(1 - k\lambda) \end{aligned}\right\} \qquad k \geq 0.$$

In particular, $\Sigma_1^* = 1$, $\Sigma_2^* = 2$, i.e. independent of $\lambda$. Closure occurs when $\Phi = \Sigma = K(0 < K < \infty)$. Therefore, if $0 < K \leq 1$, closure commences immediately from the impact end and if $1 < K \leq 2$ from the fixed end after the passage of the first wave. These results are independent of $\lambda$, so we now need consider only $2 < K < \infty$. The case $\lambda = 0$ was discussed in the previous section and it was shown that closure always took place.

*Case* $\lambda \geq 1/2$.

Here $\Sigma_r^* \leq 2$, $(r \geq 1)$, since

$$(\Sigma_{2k+2}^* - \Sigma_{2k+1}^*) = (\Sigma_{2k+1}^* - \Sigma_{2k}^*) = (1 - 2k\lambda), \qquad (k \geq 1). \tag{12}$$

Hence if $K > 2$ no closure occurs.

*Case* $\quad \dfrac{1}{2(p + 1)} \leq \lambda < \dfrac{1}{2p}, p \geq 1,$ *integer*.

Here $\Sigma_r^*$ increases strictly up to $\Sigma_{2p+2}^*$ and $\Sigma_r^* \leq \Sigma_{2p+2}^*$, all $r$. [See (12)].

But $\Sigma_{2p+2}^* = 2(p + 1)(1 - p\lambda)$. Hence, if $K > 2(p + 1)(1 - p\lambda)$, no closure occurs. If $2 < K \leq 2(p + 1)(1 - p\lambda)$, then closure does occur. In this case there is an integer $q$ such that $2 \leq q \leq (2p + 1)$ and $\Sigma_q^* < K \leq \Sigma_{q+1}^*$. Hence closure commences a $T = q + 0$, and from the impact end if $q$ is even, from the fixed end if $q$ is odd.

We now determine under what conditions complete closure of the spring occurs. The distance moved by the impact end is given by $y = v_0 t - \frac{1}{2}at^2$, whence $t = 1/a\{v_0 - \sqrt{v_0^2 - 2ay}\}$. Complete closure occurs if, and only if, $y$ attains the value $l\epsilon_{max}$, i.e. if, and only if, $v_0^2 - 2al\epsilon_{max} \geq 0$. This condition may be written in the form $2\lambda K \leq 1$.

*Closure from the fixed end.* We consider here the case where closure commences from the end $X = 1$ of the spring at $T = (2k + 1) + 0$. Hence we have

$$\{(2k + 1) - 2k^2\lambda\} < K \leq \{(2k + 2) - 2k(k + 1)\lambda\}, \qquad \text{and} \qquad (1 - 2k\lambda) > 0.$$

The closure wave has a velocity not less than $c_0$ and we suppose that the wave front has reached $X = (1 - \chi)$ at $T = (2k + 1) + \tau$. The configuration in the $(X,T)$-plane is shown in Fig. 5 (for the case $k = 0$), as before the shaded portion depicting coil closure.

The closed portion of the spring $(1 - \chi) \leq X \leq 1$ is at rest, so the stress throughout is constant and equal to that at $X = (1 - \chi)$. Now ahead of the closure wave front

$\Sigma = \Phi$ and Eq. (11) gives $\Delta\Sigma = \Omega^2\Delta\Phi$. Hence closure ceases when $\Omega = 1$, the coils then barely coming into contact. ($\Omega = d\chi/d\tau$).

Just ahead of the closure wave front we have

$$\Sigma = \Phi = \{(2k + 1)(1 - \lambda\tau) - (2k^2 + 2k + 1)\lambda + \lambda(1 - \chi)\}$$

and

$$V = \{(1 - \lambda\tau) - (2k + 1)\lambda\chi\}.$$

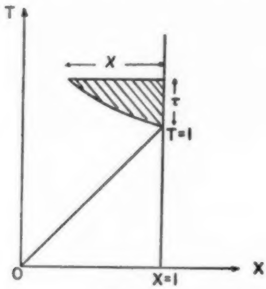


FIG. 5.

Just after the wave front has passed, $\Phi = K$, $V = 0$. Substituting in the equation $|\Delta V| = \Omega\Delta\Phi$, putting $\Omega = d\chi/d\tau$ and integrating we obtain

$$\lambda\{\chi^2 + 2(2k + 1)\chi\tau + \tau^2\} - 2\tau + 2\{K - (2k + 1) + 2k(k + 1)\lambda\}\chi = 0 \tag{13}$$

$$(\tau = 0 \quad \text{when} \quad \chi = 0).$$

Equation (13) is that of a conic in the $(X,T)$-plane. If $k = 0$ the conic is a parabola and if $k \geq 1$ a hyperbola.

*Case* $k = 0$. Closure commences at $T = 1 + 0$, $1 < K \leq 2$. The equation of the parabola is

$$\lambda(\chi + \tau)^2 - 2\tau + 2(K - 1)\chi = 0. \tag{14}$$

This may be written in the form

$$\lambda\left\{U - \frac{(2 - K)}{2\sqrt{2}\,\lambda}\right\}^2 = \frac{K}{\sqrt{2}}\left\{V + \frac{(2 - K)^2}{4K\lambda\sqrt{2}}\right\}$$

where

$$U = \frac{1}{\sqrt{2}}(\chi + \tau), \qquad V = \frac{1}{\sqrt{2}}(\tau - \chi).$$

The parabola is shown in Fig. 6, with $K < 2$. Its axis is parallel to the $V$-axis, and its vertex $A$ is at $U = (2 - K)/2\sqrt{2}\,\lambda$, $V = -(2 - K)^2/4\sqrt{2}\,K\lambda$. The tangent to the parabola at $A$ is parallel to the $U$-axis and so at $A$ we have $d\chi/d\tau = 1$. If $\chi \geq 1$ at $A$ then complete closure occurs, while if $\chi < 1$ only partial closure occurs. But at $A$

$$\chi = \frac{1}{\sqrt{2}}(U - V) = (4 - K^2)/8K\lambda.$$

Hence if $K^2 \leq 4(1 - 2\lambda K)$ complete closure of the spring occurs on this closure wave, while only partial closure occurs if $K^2 > 4(1 - 2\lambda K)$. Now $1 < K \leq 2$. Thus we obtain the necessary and sufficient conditions for complete closure in the form

$$\begin{cases} 0 \leq \lambda < 3/8 \\ 1 < K \leq 2\{\sqrt{4\lambda^2 + 1} - 2\lambda\} \end{cases} \quad \text{or} \quad \begin{cases} 1 < K \leq 2 \\ \lambda \leq (4 - K^2)/8K \end{cases} \tag{15}$$

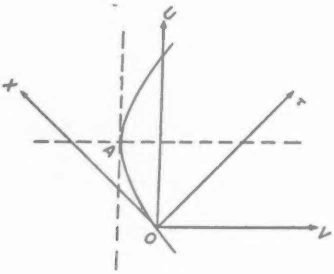

FIG. 6.

The case $\lambda = 0$ is included here and it is seen that if $K = 2$ this is the only case in which complete closure can take place. From Eq. (14), $\{1 - \lambda(\chi + \tau)\} = +\sqrt{1 - 2\lambda K\chi}$, since $\tau = 0$ when $\chi = 0$. The stress behind the closure wave $(1 - \chi \leq X \leq 1)$ is

$$\Sigma = \frac{K\{1 - \lambda(\chi + \tau)\}}{[K - \{1 - \lambda(\chi + \tau)\}]} = \frac{K\sqrt{1 - 2\lambda K\chi}}{\{K - \sqrt{1 - 2\lambda K\chi}\}}.$$

As $\chi$ increases this decreases. We also find that

$$\Omega = \frac{d\chi}{d\tau} = \frac{\sqrt{1 - 2\lambda K\chi}}{\{K - \sqrt{1 - 2\lambda K\chi}\}}.$$

The case of partial closure after the first wave has been considered. Here $1 < K \leq 2$, $K^2 + 8\lambda K - 4 > 0$. Closure ceases when

$$\chi = \frac{(4 - K^2)}{8\lambda K} = \chi^* \quad \text{and} \quad \tau = \frac{(3K - 2)(2 - K)}{8\lambda K} = \tau^*.$$

It may be shown that the spring then starts opening up at the constant rate $c_0$ (relative to the unstrained spring). If, moreover, closure does not commence from the impact end when the discontinuity wave is reflected there, then no more closures occur and the configuration in the $(X,T)$-plane is as shown in Fig. 7. The condition for no further closure is $K^2 - 8\lambda K + 4 \leq 0$, so together with the above conditions we have

$$\begin{cases} \frac{1}{2} \leq \lambda < 5/8 \\ 2\{2\lambda - \sqrt{4\lambda^2 - 1}\} \leq K \leq 2 \end{cases} \quad \text{or} \quad \begin{cases} \lambda \geq 5/8 \\ 1 < K \leq 2, \end{cases} \quad \text{or} \quad \begin{cases} 1 < K \leq 2 \\ \lambda \geq \dfrac{4 + K^2}{8K}. \end{cases} \tag{16}$$

If $K^2 > 4\,|\,1 - 2\lambda K\,|$, then partial closure occurs from the fixed end after the first wave and closure commences from the impact end after the second wave. Complete closure may, or may not, eventually occur (according as $2\lambda K \leq 1$, or $2\lambda K > 1$, respectively).

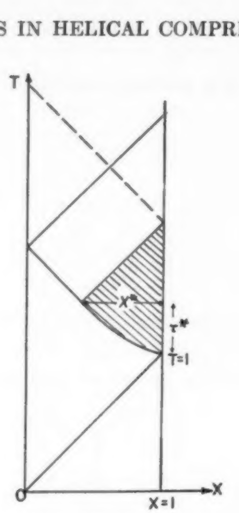


Fig. 7.

*Closure from the impact end.* We consider here the case in which closure commences from the end $X = 0$ of the spring at time $T = 2k + 0$, $(k \geq 0)$. Hence we have

$$2k\{1 - (k - 1)\lambda\} < K \leq \{(2k + 1) - 2k^2\lambda\}, \qquad \text{and} \qquad (1 - 2k\lambda) > 0.$$

The closure wave has a velocity not less than $c_0$ and we suppose that the wave front has reached $X = \chi$ at $T = 2k + \tau$. The closed portion of the coil is moving as a rigid body with the velocity of the impact end and hence it is being decelerated at a constant rate. The configuration in the $(X,T)$-plane is shown in Figs. 8 and 9 for the cases $k = 0$, $k = 1$, respectively.

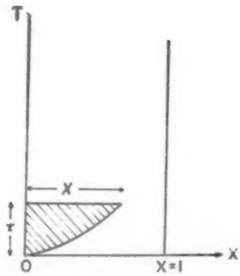


Fig. 8.

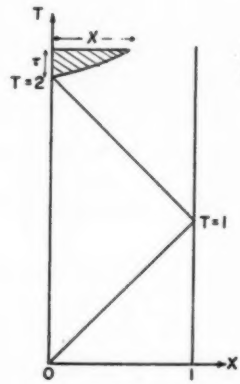


Fig. 9.

By use of Eq. (11) we find $\chi$, $\tau$ are connected by the relation

$$\lambda\tau^2 + 4k\lambda\chi\tau + 2\{K - 2k(1 - k\lambda)\}\chi - 2\tau = 0. \tag{17}$$

If $k = 0$ this is the equation of a parabola and if $k \geq 1$ that of a hyperbola. We consider the case $k = 0$.

Since the portion $0 \leq X \leq \chi$ is moving as a rigid body with uniform retardation $a$ we obtain

$$\Sigma = \Sigma|_{X=\chi} - \lambda(\chi - X), \qquad 0 \leq X \leq \chi. \tag{18}$$

Hence the stress in the closed portion decreases as $X$ decreases. From Eq. (11) we find

$$\Sigma|_{X=\chi} = \frac{1}{K}(1 - \lambda\tau)^2 = \frac{1}{K}(1 - 2\lambda K\chi),$$

using Eq. (17). Hence at a pt. in the closed portion $\Sigma$ decreases as $\chi$ increases. Complete closure takes place if, and only if, $(K^2 + 3\lambda K - 1) \leq 0$. We also have the condition $0 < K \leq 1$. Hence we obtain the necessary and sufficient conditions for complete closure in the form

$$\begin{cases} \lambda \geq 0 \\ 0 < K \leq \frac{1}{2}\{\sqrt{9\lambda^2 + 4} - 3\lambda\}, \end{cases} \qquad \text{or} \qquad \begin{cases} 0 < K \leq 1 \\ 0 \leq \lambda \leq \dfrac{(1-K)}{3K}. \end{cases} \tag{19}$$

**4. Mass attached to impact end of spring.** In this section we consider the case in which a mass $M$ is attached to the end $x = 0$ of the spring. The spring is initially at rest and unstrained, the axis of the spring being horizontal, when the attached mass is given an impulsive velocity $v_0$ at time $t = 0$.

The equation of motion for the mass $M$ is

$$\frac{dW}{dT} = -\mu S, \tag{20}$$

where $W = V|_{x=0}$, $S = \Sigma|_{x=0}$ and $\mu = ml/M$. Until closure occurs the configuration in the $(X,T)$-plane is as shown in Fig. 3. A wave of velocity, stress and strain discontinuity is propagated along the spring with constant speed $c_0$, commencing at the end $x = 0$ at time $t = 0$ and being reflected alternately at the ends.

Using Eqs. (4), (5) and (20), we find

$$\begin{aligned} &\text{In region 0: } \Sigma = V = 0 \\ &\text{In region 1: } \Sigma = V = e^{-\mu(T-X)} \\ &\text{In region 2: } \Sigma = 2e^{-\mu(T-1)} \cosh\{\mu(X-1)\}, \\ &\qquad\qquad\quad V = 2e^{-\mu(T-1)} \sinh\{\mu(X-1)\}. \end{aligned} \tag{21}$$

Since, in region 1, $T \geq X$, we have $\Sigma \leq 1$. ($\mu > 0$.) Hence, if $0 < K \leq 1$ closure commences from the impact end immediately and if $K > 1$ no closures occur in $0 \leq T < 1$. For $1 \leq T < 2$, if $1 < K \leq 2$ closure commences from the fixed end at $T = 1$ and if $K > 2$ no closures occur. If $2 < K \leq (2 + e^{-2\mu})$ closure commences from the gun end at $T = 2$. We consider these three closures in detail.

*Closure from fixed end after first wave.* $1 < K \leq 2$. Closure commences at $T = 1$. We suppose that at $T = 1 + \tau$ the closure wave has moved along the spring to $X =$

$(1 - \chi)$ the portion $(1 - \chi) \le X \le 1$ being closed and at rest. The configuration in the $(X,T)$-plane is as shown in Fig. 5. Now just ahead of the closure wave front we have $\Phi = V = \Sigma = e^{-\mu(\tau+\chi)}$. Setting $\Omega = d\chi/d\tau$ we obtain from Eq. (11)

$$e^{-\mu(\tau+\chi)} = \frac{d\chi}{d\tau}\{K - e^{-\mu(\tau+\chi)}\}.$$

Integrating,

$$K\mu\chi + e^{-\mu(\chi+\tau)} = 1 \tag{22}$$

since initially $\chi = 0 = \tau$. Hence

$$\tau = \frac{1}{\mu}\log\left\{\frac{1}{1 - K\mu\chi}\right\} - \chi.$$

Also, $\Omega = (1 - K\mu\chi)/\{(K - 1) + K\mu\chi\}$ which decreases as $\chi$ increases. We find $\Sigma = K(1 - K\mu\chi)/\{(K - 1) + K\mu\chi\}$, for $(1 - \chi) \le X \le 1$. Complete closure occurs on this wave if, and only if, $(1 - K\mu) \ge \{(K - 1) + K\mu\}$, i.e., $K(1 + 2\mu) \le 2$. But $1 < K \le 2$. Therefore,

$$\begin{cases} 1 < K \le 2 \\ \mu \le \dfrac{(2 - K)}{2K} \end{cases} \quad or \quad \begin{cases} 0 \le \mu < 1/2 \\ 1 < K \le \dfrac{2}{(1 + 2\mu)}. \end{cases}$$

If complete closure occurs then the time taken is given by $(1 + \tau) = 1/\mu \log(1/(1 - K\mu)) = T'$. But during the period $0 \le T \le T'$, $W = e^{-\mu T}$. Therefore $\int_0^{T'} W\,dT = K$ so that the gun end has moved through a distance $l\epsilon_{\max}$, which is correct.

If complete closure does not occur then closure ceases when $\Omega = 1$, i.e., when $\chi = (2 - K)/2K\mu = \chi^*$ and then $\tau = 1/2\mu\,\{2\log(2/K) + 1 - (2/K)\} = \tau^*$.

*Closure from impact end at* $T = 2k(k \ge 0)$. Let us suppose that closure commences (for the first time) at the impact end at $T = 2k$. With the usual notation put $T = 2k + \tau$, and let $X = \chi$ be the position of the closure wave front. Let the stress and velocity just ahead of the closure wavefront be given by $\Phi = \Sigma = \Sigma(\chi, \tau)$, $V = V(\chi, \tau)$. Since no previous closures have occurred Eq. (3) holds in this region ahead of the closure wavefront. Hence we obtain

$$\frac{\partial V}{\partial \tau} + \frac{\partial \Sigma}{\partial \chi} = 0, \qquad \frac{\partial \Sigma}{\partial \tau} + \frac{\partial V}{\partial \chi} = 0.$$

$W = W(\tau)$ gives the velocity of the impact end and $S = \Sigma\,|_{X=0}$. From Eq. (11) we find

$$\{W - V(\chi, \tau)\} = \frac{d\chi}{d\tau}\{K - \Sigma(\chi, \tau)\}. \tag{23}$$

and

$$\Sigma|_{X=\chi} = \Sigma(\chi, \tau) + \frac{d\chi}{d\tau}\{W - V(\chi, \tau)\}. \tag{24}$$

But, since the portion $0 \leq X \leq \chi$ is closed and moving as a rigid body,

$$\Sigma = \Sigma|_{X=\chi} + (\chi - X)\frac{dW}{d\tau}, \qquad 0 \leq X \leq \chi$$

and in particular

$$S = \Sigma|_{X=\chi} + \chi\frac{dW}{d\tau}.$$

Hence

$$S = \left(\chi\frac{dW}{d\tau} + W\frac{d\chi}{d\tau}\right) + \left\{\Sigma(\chi, \tau) - \frac{d\chi}{d\tau}V(\chi, \tau)\right\}.$$

But, from Eq. (20), $dW/d\tau + \mu S = 0$. Hence

$$\frac{d}{d\tau}\{W(1 + \mu\chi)\} + \mu\left\{\Sigma(\chi, \tau) - V(\chi, \tau)\frac{d\chi}{d\tau}\right\} = 0.$$

Let $F(\chi, \tau)$ satisfy the conditions

$$-\frac{\partial F}{\partial \tau} = \mu\Sigma, \qquad \frac{\partial F}{\partial \chi} = \mu V, \qquad F(0, 0) = W(0),$$

in virtue of the equation $\partial V/\partial\tau + \partial\Sigma/\partial\chi = 0$.

Then we obtain on integration

$$W(1 + u\chi) = F(\chi, \tau). \tag{25}$$

Substituting for $W$ from Eq. (23) we find that

$$(1 + \mu\chi)\{K - \Sigma(\chi, \tau)\}\frac{d\chi}{d\tau} + \{(1 + \mu\chi)\cdot V(\chi, \tau) - F(\chi, \tau)\} = 0. \tag{26}$$

But

$$\frac{\partial}{\partial\tau}[(1 + \mu\chi)\{K - \Sigma(\chi, \tau)\}] - \frac{\partial}{\partial\chi}\{(1 + \mu\chi)\cdot V(\chi, \tau) - F(\chi, \tau)\}$$

$$= -(1 + \mu\chi)\cdot\frac{\partial\Sigma}{\partial\tau} - \mu V + \frac{\partial F}{\partial\chi} - (1 + \mu\chi)\frac{\partial V}{\partial\chi}$$

$$= \left(\frac{\partial F}{\partial\chi} - \mu V\right) - (1 + \mu\chi)\cdot\left(\frac{\partial\Sigma}{\partial\tau} + \frac{\partial V}{\partial\chi}\right) = 0.$$

Hence the differential equation integrates immediately.

*Case* $k = 0$. $0 < K \leq 1$. Closure commences from the impact end immediately. The configuration in the $(X,T)$-plane is as shown in Fig. 8. Here $\Sigma(\chi, \tau) = V(\chi, \tau) = 0$, and $W(0) = 1$. Therefore $F(\chi, \tau) = 1$. Hence, from Eq. (25), $W(1 + \mu\chi) = 1$ and from Eq. (26), $K(1 + \mu\chi)\,d\chi/d\tau = 1$. Therefore $K\chi(2 + \mu\chi) = 2\tau$, since $\chi = 0$ when $\tau = 0$.

This is the equation of a parabola whose axis is parallel to the $T$-axis. We find that

$$\Omega = \frac{d\chi}{d\tau} = \frac{1}{K(1 + \mu\chi)} \qquad \text{and} \qquad S = \frac{1}{K(1 + \mu\chi)^3}.$$

Also we have

$$\Sigma = \frac{(1 + \mu X)}{K(1 + \mu\chi)^3}, \qquad 0 \leq X \leq \chi.$$

Hence in the closed portion $\Sigma$ decreases as $X$ decreases so that the coils remain in contact provided $S \geq K$. As $\chi$ increases $S$ decreases so complete closure takes place if, and only if,

$$S|_{\chi=1} \geq K, \qquad \text{i.e.} \qquad K^2(1 + \mu)^3 \leq 1. \qquad \text{but} \qquad 0 < K \leq 1.$$

Therefore,

$$\begin{cases} \mu \geq 0 \\ 0 < K \leq (1 + \mu)^{-3/2} \end{cases} \qquad \text{or} \qquad \begin{cases} 0 < K \leq 1 \\ 0 \leq \mu \leq (K^{-2/3} - 1). \end{cases}$$

If this condition is not satisfied the coils start to open up from the end $X = 0$ when

$$\chi = \frac{1}{\mu}(K^{-2/3} - 1) = \chi^*$$

and then

$$\tau = \frac{K}{2\mu}(K^{-4/3} - 1) = \tau^*.$$

*Case* $k = 1$. $2 < K \leq (2 + e^{-2\mu})$.

The configuration in the $(X,T)$-plane is as shown in Fig. 9. Here

$$\Sigma(\chi, \tau) = 2e^{-\mu(\tau+1)} \cosh \{\mu(1 - \chi)\},$$

$$V(\chi, \tau) = -2e^{-\mu(\tau+1)} \sinh \{\mu(1 - \chi)\} \qquad \text{and} \qquad W(0) = e^{-2\mu}.$$

Hence, from Eq. (25),

$$W(1 + \mu\chi) = 2e^{-\mu(\tau+1)} \cosh \{\mu(1 - \chi)\} - 1.$$

Also, from Eq. (26), we obtain on integration

$$K\mu\chi(2 + \mu\chi) + 2\mu\tau = 4\{1 - e^{-\mu(1+\tau)}[(1 + \mu\chi) \sinh \{\mu(1 - \chi)\} + \cosh \{\mu(1 - \chi)\}]\}.$$

**5. Discussion of results obtained.** This paper has been devoted to investigating the occurence of coil closure and the stresses during closure in a helical compression spring one end of which is fixed, the other being given an impulsive velocity when the spring is initially unstrained and at rest. The number $K = 1/v_0\ (c_0\epsilon_{\max})$ arises, where $v_0$ is the impact velocity and $c_0\epsilon_{\max}$ is a quantity depending on the dimensions and material of the spring. Inelastic coil on coil impact conditions have been assumed so that the coils remain in contact after closure, until separated by the elasticity of the spring. The results may, however, give a good indication as to the actual state of affairs in which bouncing of the coils occurs.

In the first instance the case in which the impulsive velocity at the impact end was maintained was considered. Here complete closure inevitably occurs and it was found that exceptionally high stresses could be set up during closure if the coils were very close together prior to the passage of the closure wave. Closure commences from either the impact or the fixed end and complete closure occurs on that wave.

Secondly, the case in which the impact end was retarded at a constant rate $a$ was considered. The number $\lambda = (a/v_0)(l/c_0)$ occurs, where $l/c_0$ is a quantity depending on the dimensions and the material of the spring. The criterion for complete closure is $2\lambda K \leq 1$, so that if complete closure of the spring is to be avoided we need $(2a/v_0)l\epsilon_{\max} > 1$. Note that $l\epsilon_{\max}$ depends on the dimensions of the spring only and not on the material. Thus it is seen that, unlike the first case, complete closure does not necessarily occur and for a given spring and prescribed impact velocity it will not occur if the retardation of the impact end is sufficiently large. Further, it is found that it is possible that no closure whatsoever will occur but that if closure does occur it commences from an end of the spring. If $0 < K \leq 1$ closure commences immediately from the impact end, whilst if $1 < K \leq 2$ closure commences from the fixed end after the passage of the first discontinuity wave, and these statements are independent of the value of $\lambda$. For $K > 2$, no closure occurs if $\lambda \geq 1/2$. If $1 < K \leq 2$ and only partial closure occurs from the fixed end after the first wave, then, if closure does not commence from the impact end after the second wave, no more closures occur at all. However, if closure does start from the impact end after the second wave complete closure of the spring does not necessarily occur.

Lastly, the case in which there was a mass attached to the impact end was considered. The number $\mu$, which is equal to the ratio of the total weight of the spring to the weight of the attached mass, occurs. This will, in general, be small. If $0 < K \leq 1$ closure commences immediately from the impact end and unless $K$ is fairly close to 1 complete closure will occur on this wave. If $1 < K \leq 2$ closure commences from the fixed end after the first wave and complete closure will occur on this wave unless $K$ is fairly close to 2 ($\mu$ being small). However, if $\mu \geq 1/2$, complete closure will not occur on this wave, however close $K$ is to 1. It is noted that $\mu = 0$ in this case and $\lambda = 0$ in the second case both correspond to the first case.

We will now discuss the stresses in the spring. Before closure occurs the stress, or nominal compressive force, is proportional to the nominal compressive strain and less than the quantity $mc_0v_0K$. When, however, coils of the spring come into contact, and remain so, the compressive force is equal to $mc_0v_0\Sigma$ where $\Sigma > K$. In the first case the compressive force in the closed portion of the spring remains constant while closure is taking place and, as previously mentioned, we may have a large compressive force. The time taken for the closure wave to traverse the length of the spring is correspondingly small, though. In the other two cases the value of $\Sigma$ does not remain constant in the closed portion of the spring while closure is taking place. When closure commences from the fixed end $\Sigma$ is the same throughout the closed portion at any given instant, but it decreases as the closure wavefront passes only the spring. When closure commences from the impact end the value of $\Sigma$ at a given position in the closed portion of the spring again decreases as the closure wavefront passes along the spring. In addition, however, the value of $\Sigma$ decreases uniformly as we pass along the spring from just behind the closure wavefront to the impact end, this uniform rate of decrease being independent of the time in the second case.

## APPENDIX

$$\begin{cases} \text{In region } 2r, \quad (r \geq 0) \\ \quad V = 2r\lambda(X - 1) \\ \quad \Sigma = 2r\{(1 - \lambda T) + r\lambda\} \\ \text{In region } (2r + 1), \quad (r \geq 0) \\ \quad V = \{(1 - \lambda T) + (2r + 1)\lambda X\} \\ \quad \Sigma = \{(2r + 1)(1 - \lambda T) + \lambda X + 2r(r + 1)\lambda\} \end{cases}$$

$$\begin{cases} \text{At} \quad T = 2k + \tau, \quad (k \geq 0), \quad 0 < \tau \leq 1, \quad 0 \leq X \leq \tau \\ \quad V = \{(1 - \lambda\tau) - 2k\lambda + (2k + 1)\lambda X\} \\ \quad \Sigma = \{(2k + 1)(1 - \lambda\tau) - 2k^2\lambda + \lambda X\} \\ \text{At} \quad T = 2k + \tau, \quad (k \geq 0), \quad 0 \leq \tau < 1, \quad \tau < X \leq 1 \\ \quad V = 2k\lambda(X - 1) \\ \quad \Sigma = 2k\{(1 - \lambda\tau) - k\lambda\} \\ \text{At} \quad T = (2k + 1) + \tau, \quad (k \geq 0), \quad 0 \leq \tau < 1, \quad 0 \leq X < (1 - \tau) \\ \quad V = \{(1 - \lambda\tau) + (2k + 1)\lambda(X - 1)\} \\ \quad \Sigma = \{(2k + 1)(1 - \lambda\tau) - (2k^2 + 2k + 1)\lambda + \lambda X\} \\ \text{At} \quad T = (2k + 1) + \tau, \quad (k \geq 0), \quad 0 < \tau \leq 1, \quad (1 - \tau) \leq X \leq 1 \\ \quad V = 2(k + 1)\lambda(X - 1) \\ \quad \Sigma = 2(k + 1)\{(1 - \lambda\tau) - k\lambda\} \end{cases}$$

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# ON FINITE TWISTING AND BENDING OF CIRCULAR RING SECTOR PLATES AND SHALLOW HELICOIDAL SHELLS*

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**1. Introduction.** In the following we consider a thin circular ring sector plate under the action of two equal and opposite forces perpendicular to the plane of the plate, along the axis through the center of the ring (Fig. 1). The ring sector plate may be con-

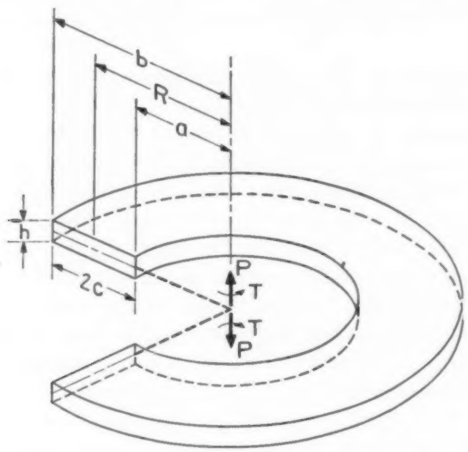


FIG. 1.

sidered as part of a winding of a close-coiled helical spring. The problem of the stress distribution in the twisted ring sector with rectangular cross section has first been considered by J. H. Michell[1] as a problem of three dimensional elasticity. A solution of this problem by means of the theory of thin plates will bear the same relation to Michell's solution as Kelvin and Tait's solution for the torsion of a rectangular plate bears to St. Venant's solution for the torsion of a beam with rectangular cross section.

The reason for the present note is the further observation that by treating the problem as a plate problem it becomes possible to analyse non-linear effects in a relatively simple manner by making use of the equations for finite deflections of thin plates.

In addition to the problem of non-linear effects for an originally flat plate we also consider the corresponding problem for a shallow helicoidal shell, thereby obtaining information concerning the influence of initial pitch on stresses and deformations in the spring.

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[1]J. H. Michell, Proc. London Math. Soc., **31**, 140-141 (1889).

In the consideration of both these problems one is led to a study of the simultaneous action of the pair of axial forces described above and of a pair of couples the axes of which coincide with the direction of these forces (Fig. 1). We determine in particular of what magnitude the couples have to be in order to prevent the association of circumferential displacements of the points of the ring plate with the axial displacements caused by the axial forces. We find that if these circumferential displacements are prevented the effect of non-linearity and the influence of initial pitch are much more pronounced than in the absence of the couples. We also encounter a problem of instability of the ring sector plate which is associated with the presence of the couples.

A further result which we find is to the effect that non-linearity is relatively more pronounced in regard to the magnitude of stresses than in regard to the axial force-displacement relation. A similar result is known for the problem of finite torsion of thin rectangular plates[2].

It seems worth noting that the present developments may also be considered as a contribution to a non-linear theory of dislocations.

Apart from the question of method the present treatment of the problems of the ring sector differs from that by means of the theory of thin rods[3] in the following respects.

Account is taken of a non-linear effect which is significant if the ratio of width of cross section to thickness of cross section is sufficiently large. For the helicoidal shell this effect is found to be present even in the range of applicability of the linearized theory. In the theory of torsion of rectangular plates this effect manifests itself through axial normal stresses proportional to the square of the angle of twist. These normal stresses have no axial resultant or bending couple but give rise to a twisting couple which is proportional to the cube of the angle of twist[2]. A corresponding result is here obtained for the problem of the ring sector.

While the results of the theory of thin rods are based on the assumption that the ratio of width of cross section to radius of center line of cross section is sufficiently small the present results hold for all possible values of this ratio.

**2. Differential equations and boundary conditions of the problem.** The differential equations for finite bending of thin plates of uniform thickness are in polar coordinate form[4]

$$D\nabla^2\nabla^2 w = p(r, \theta) + \frac{1}{r}\frac{\partial}{\partial r}\left(rN_r\frac{\partial w}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(N_\theta\frac{1}{r}\frac{\partial w}{\partial \theta}\right)$$

$$+ \frac{1}{r}\frac{\partial}{\partial r}\left(N_{r\theta}\frac{\partial w}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(N_{r\theta}\frac{\partial w}{\partial r}\right), \tag{1}$$

$$\nabla^2\nabla^2 F = Eh\left\{\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right]^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\right\}, \tag{2}$$

where $w$ is the transverse deflection, $F$ Airy's stress function, and $D = Eh^3/12(1 - \nu^2)$.

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[2]S. Timoshenko, *Strength of materials*, Part 2, Van Nostrand, p. 301; A. E. Green, Proc. Roy. Soc. London (A) **154**, 430-455 (1936); **161**, 197-220 (1937).

[3]See A. E. H. Love, *Treatise on the mathematical theory of elasticity*, 4th Ed., Cambridge 1934, pp. 414-417 for reference to work by Kirchhoff, Kelvin and Tait, St. Venant, and Perry.

[4]K. Federhofer, Z. Angew. Math. Mech. **25/27**, 20 (1947).

Stress resultants and couples are given as follows

$$N_\theta = \frac{\partial^2 F}{\partial r^2}, \qquad N_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}, \qquad N_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right), \tag{3}$$

$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right), \qquad M_\theta = -D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right),$$

$$M_{r\theta} = -(1-\nu)D\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right), \tag{4}$$

$$V_r = -D\frac{\partial \nabla^2 w}{\partial r}, \qquad V_\theta = -D\frac{\partial \nabla^2 w}{r\,\partial \theta}, \tag{5}$$

$$R_r = V_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} + N_r\frac{\partial w}{\partial r} + N_{r\theta}\frac{1}{r}\frac{\partial w}{\partial \theta},$$

$$R_\theta = V_\theta + \frac{\partial M_{r\theta}}{\partial r} + N_\theta\frac{1}{r}\frac{\partial w}{\partial \theta} + N_{r\theta}\frac{\partial w}{\partial r}\,. \tag{6}$$

In addition to this there occur concentrated corner forces of magnitude $2M_{r\theta}$ at corners formed by lines $r = \text{const.}$ and $\theta = \text{const.}$

Let $r = a$ and $r = b$ be the inner and outer circular edges of the plate and let $\theta = 0$ and $\theta = \alpha$ be the radial edges of the plate (Fig. 1). We prescribe the following boundary conditions, assuming for the present that the couples $T$ indicated in Fig. 1 are absent,

$$r = a, b; \quad N_r = N_{r\theta} = R_r = M_r = 0 \tag{7}$$

$$\theta = 0, \alpha; \begin{cases} \int_a^b N_\theta\, dr = \int_a^b N_{r\theta}\, dr = \int_a^b rN_\theta\, dr = \int_a^b M_\theta\, dr = 0 \\ \int_a^b R_\theta\, dr + 2(M_{r\theta})_a - 2(M_{r\theta})_b = P \\ \int_a^b rR_\theta\, dr + 2(rM_{r\theta})_a - 2(rM_{r\theta})_b = 0 \end{cases} \tag{8}$$

The boundary conditions at the edges $\theta = \text{const.}$ are taken in such form that a semi-inverse procedure of solution becomes possible.

**3. Solution of the boundary value problem.** The fact that the solution to be obtained should give stress resultants and couples which are independent of the polar angle $\theta$ and a transverse displacement $w$ which is proportional to $\theta$ and knowledge of the corresponding solution in three-dimensional linear elasticity suggests the following form of $w$ and $F$,

$$w = k\theta, \qquad F = F(r), \tag{9}$$

Substitution of (9) in the differential equations (1) and (2), with $p(r, \theta) = 0$, shows that (1) is identically satisfied and (2) becomes

$$\nabla^2\nabla^2 F = Eh\frac{k^2}{r^4}\,. \tag{10}$$

From (10),

$$F = \tfrac{1}{8}Ehk^2[(\ln r)^2 + A \ln r + Br^2 + Cr^2 \ln r], \tag{11}$$

where $A$, $B$, $C$ are constants of integration and an unessential additive constant has been omitted.

Stress resultants and stress couples are now

$$\begin{aligned}
N_r &= \frac{1}{8} Ehk^2\left[2\,\frac{\ln r}{r^2} + \frac{A}{r^2} + 2B + C(1 + 2 \ln r)\right],\\
N_\theta &= \frac{1}{8} Ehk^2\left[2\,\frac{1 - \ln r}{r^2} - \frac{A}{r^2} + 2B + C(3 + 2 \ln r)\right],\\
N_{r\theta} &= 0, \qquad V_r = 0, \qquad V_\theta = 0,\\
M_r &= 0, \qquad M_\theta = 0, \qquad M_{r\theta} = (1 - \nu)D\,\frac{k}{r^2},\\
R_r &= 0, \qquad R_\theta = -2(1 - \nu)D\,\frac{k}{r^3} + N_\theta\,\frac{k}{r}\,.
\end{aligned} \tag{12}$$

Determination of the constants of integration $A$, $B$, $C$ and $k$ is effected by means of the boundary conditions (7) and (8), some of which are seen to be satisfied identically. The relevant remaining conditions are the first of equations (7) and the third and fifth of Eqs. (8). We have from (7)

$$\begin{aligned}
\frac{A}{a^2} + 2B + C(1 + 2 \ln a) &= -2\,\frac{\ln a}{a^2}\,,\\
\frac{A}{b^2} + 2B + C(1 + 2 \ln b) &= -2\,\frac{\ln b}{b^2}\,.
\end{aligned} \tag{13}$$

The third of Eqs. (8) may be transformed as follows if account is taken of the fact that $F'(b) = F'(a) = 0$,

$$\int_a^b rN_\theta\, dr = \int_a^b rF''\, dr = (rF')_a^b - \int_a^b F'\, dr = F(b) - F(a) = 0.$$

We have then

$$\begin{aligned}
A \ln (b/a) + B(b^2 - a^2) + C(b^2 \ln b - a^2 \ln a) &= -(\ln b)^2 + (\ln a)^2\\
&= -\ln (b/a) \ln ab.
\end{aligned} \tag{14}$$

Equation (13) and (14) serve to determine $A$, $B$ and $C$. The remaining fifth of Eqs. (8) is to be used for the determination of the relation between force $P$ and deflection $k$. We may write

$$\int_a^b N_\theta r^{-1}\, dr = \int_a^b F''r^{-1}\, dr = (F'r^{-1})_a^b + \int_a^b F'r^{-2}\, dr = \int_a^b N_r r^{-1}\, dr.$$

Therewith, we have from this fifth of Eqs. (8),

$$(1-\nu)Dk\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\frac{1}{8}Ehk^3\left[\frac{1+2\ln a}{2a^2}-\frac{1+2\ln b}{2b^2}+\frac{A}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\right.$$

$$\left.+\,2B\ln\frac{b}{a}+C\left(\ln\frac{b}{a}+(\ln b)^2-(\ln a)^2\right)\right]=P. \tag{15}$$

After some transformations in which we make use of the following formulas

$$A\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=2C\ln\frac{b}{a}+2\left(\frac{\ln b}{b^2}-\frac{\ln a}{a^2}\right),$$

$$2B(b^2-a^2)=-C[b^2-a^2+2(b^2\ln b-a^2\ln a)]-2\ln\frac{b}{a}, \tag{16}$$

$$C\left\{\frac{2a^2b^2}{b^2-a^2}\left(\ln\frac{b}{a}\right)^2-\frac{b^2-a^2}{2}\right\}=\ln\frac{b}{a}\left\{1-\frac{b^2+a^2}{b^2-a^2}\ln\frac{b}{a}\right\},$$

$$a=R-c,\qquad b=R+c,\qquad E=2(1+\nu)G. \tag{17}$$

Eq. (15) appears in the form

$$k\left[1+\frac{3(1+\nu)}{4}f_1\left(\frac{c}{R}\right)\left(\frac{k}{h}\right)^2\right]=\frac{3PR^3}{2Gch^3}\left[1-\left(\frac{c}{R}\right)^2\right]^2, \tag{18}$$

where

$$f_1(x)=\frac{\left\{\left[1-\left(\frac{1-x^2}{2x}\ln\frac{1+x}{1-x}\right)^2\right]^2-\left(\frac{1-x^2}{2x}\ln\frac{1+x}{1-x}\right)^2\left[\frac{1+x^2}{2x}\ln\frac{1+x}{1-x}-1\right]^2\right\}}{\left\{1-\left(\frac{1-x^2}{2x}\ln\frac{1+x}{1-x}\right)^2\right\}}. \tag{19}$$

For sufficiently small values of $x$, practically up to $x \sim 1/4$, we have

$$f_1(x)\sim\frac{16}{45}x^4,\qquad x\lesssim\frac{1}{4} \tag{20a}$$

For larger values of $x$ we have the limiting relation

$$\lim_{x\to1}f_1(x)=1 \tag{20b}$$

Equations (18) and (20) indicate the way in which the nonlinear effect depends on deformation and dimensions of the plate. For wide plates for which $1-x\ll1$ we have the usual result that for linearity the ratio of deflection per unit of circumferential angle to plate thickness must be small compared to unity. As the plate gets narrower larger and larger deflections lie within the linear range, the condition for linearity being that $(k/h)(c/R)^2$ be somewhat smaller than unity.

A further question of interest concerns the magnitude of the stresses in the plate. As we have neglected the effect of transverse shear deformation we cannot determine the magnitude of the *transverse* shearing stresses. But we can determine, within the accuracy

of the results of plate theory the magnitude of the horizontal shearing stress $\tau_{r\theta} = 6M_{r\theta}/h^2$. We have from (12)

$$\tau_{r\theta} = (1 - \nu)D \frac{6k}{r^2h^2} \tag{21a}$$

and from this

$$\tau_{r\theta}(a) = \tau_{r\theta,\max} = \frac{E}{2(1 + \nu)} \frac{kh}{a^2} . \tag{21b}$$

In the non-linear range we have in addition a stress $\sigma_\theta = N_\theta/h$. We limit ourselves here to a consideration of this stress along the edges $r = a$ and $r = b$ of the plate. We have from (12), taking account of the fact that $N_r(a) = N_r(b) = 0$,

$$\begin{aligned} \sigma_\theta(a) &= \frac{1}{8} Ek^2\left\{\frac{2}{a^2} - 4 \frac{\ln b/a}{b^2 - a^2} + C\left[2 - \frac{4b^2}{b^2 - a^2} \ln \frac{b}{a}\right]\right\}, \\ \sigma_\theta(b) &= \frac{1}{8} Ek^2\left\{\frac{2}{b^2} - 4 \frac{\ln b/a}{b^2 - a^2} + C\left[2 - \frac{4a^2}{b^2 - a^2} \ln \frac{b}{a}\right]\right\}, \end{aligned} \tag{22}$$

where $C$ is given by (16).

Equations (22) may be written in the alternate form

$$\sigma_\theta(a) = \frac{1}{4} \frac{Ek^2}{R^2} \frac{f_2(c/R)}{(1 - c/R)^2} , \qquad \sigma_\theta(b) = \frac{1}{4} \frac{Ek^2}{R^2} \frac{f_3(c/R)}{(1 + c/R)^2} , \tag{23}$$

where

$$\begin{aligned} f_2(x) &= 1 - \frac{(1 - x)^2}{2x} \ln \frac{1 + x}{1 - x} + CR^2(1 - x)^2\left[1 - \frac{(1 + x)^2}{2x} \ln \frac{1 + x}{1 - x}\right], \\ f_3(x) &= 1 - \frac{(1 + x)^2}{2x} \ln \frac{1 + x}{1 - x} + CR^2(1 + x)^2\left[1 - \frac{(1 - x)^2}{2x} \ln \frac{1 + x}{1 - x}\right], \\ CR^2 &= \frac{1}{2x} \ln \frac{1 + x}{1 - x} \left\{\frac{1 + x^2}{2x} \ln \frac{1 + x}{1 - x} - 1\right\}\Big/\left\{1 - \left(\frac{1 - x^2}{2x} \ln \frac{1 + x}{1 - x}\right)^2\right\}. \end{aligned} \tag{24}$$

Of the two stresses $\sigma_\theta(a)$ and $\sigma_\theta(b)$ the larger one is the stress $\sigma_\theta(a)$ at the inner edge of the plate. For the sake of comparison we may write

$$\frac{\sigma_\theta(a)}{\tau_{r\theta}(a)} = \frac{1 + \nu}{2} f_2\left(\frac{c}{R}\right) \frac{k}{h} \tag{25}$$

For sufficiently small values of $c/R$, practically for $c/R < 1/4$, we have

$$f_2\left(\frac{c}{R}\right) \sim \frac{4}{3} \left(\frac{c}{R}\right)^2, \qquad \frac{c}{R} \lesssim \frac{1}{4} \tag{26a}$$

For larger values of $c/R$ we have the limiting relation

$$\lim_{c/R \to 1} f_2\left(\frac{c}{R}\right) = 1 \tag{26b}$$

Comparison of the stress ratio (25) with the load deflection relation (18) indicates that non-linearity affects the stresses much more strongly than the deflection characteristics of the plate. As long as $c/R < 1/4$ we have for example that when $\sigma_\theta(a)/\tau_{r\theta}(a) = 1/4$ then non-linearity is responsible for a three-percent correction only to the stress displacement relation and when $\sigma_\theta(a)/\tau_{r\theta}(a) = 1/2$ then this correction amounts to only about twelve percent.

**4. Interaction between pure twisting and pure bending.** A further problem within the present context is the problem of the effect of a moment $T$ about the line of action of the force $P$ on the relation between force $P$ and deflection $k$. Differential equations and boundary conditions are as before with one exception, which consists in replacing the boundary condition $\int_a^b rN_\theta\, dr = 0$ by the condition

$$\int_a^b rN_\theta\, dr = T. \tag{27}$$

The solution for the stress function $F$ now consists of two parts, one due to non-linearity in $k$ and with $T = 0$, and the other without the non-linearity in $k$ but with $T \neq 0$. Since the first part of the solution is given in the previous section we may now restrict attention to the second part of the solution. We write for this second part

$$F = T\{A^* \ln r + B^* r^2 + C^* r^2 \ln r\}. \tag{28}$$

The boundary conditions $N_r(a) = N_r(b) = 0$ are satisfied by setting

$$\begin{aligned} A^* a^{-2} + 2B^* + C^*(1 + 2\ln a) &= 0, \\ A^* b^{-2} + 2B^* + C^*(1 + 2\ln b) &= 0. \end{aligned} \tag{29}$$

The condition $\int_a^b rN_\theta\, dr = T$ becomes

$$A^* \ln(b/a) + B^*(b^2 - a^2) + C^*(b^2 \ln b - a^2 \ln a) = -T. \tag{30}$$

From (29) and (30) follows

$$\begin{aligned} A^* \frac{b^2 - a^2}{a^2 b^2} = 2C^* \ln\frac{b}{a}, \qquad -B^* &= \left(\frac{1}{2} + \frac{b^2 \ln b - a^2 \ln a}{b^2 - a^2}\right) C^*, \\ C^* = \frac{2}{b^2 - a^2}\left[1 - \left(\frac{2ab}{b^2 - a^2}\ln\frac{b}{a}\right)^2\right]^{-1}&. \end{aligned} \tag{31}$$

A relation between $P$ and $k$ follows again from $\int_a^b R_\theta\, dr - 2[M_{r\theta}]_b^a = P$, in the form

$$(1 - \nu)Dk\frac{b^2 - a^2}{a^2 b^2} + Tk\left[\frac{A^*}{2}\frac{b^2 - a^2}{b^2 a^2} + 2B^* \ln\frac{b}{a} + C^*(1 + \ln ab)\ln\frac{b}{a}\right] = P. \tag{32}$$

Introduction of $A^*$, $B^*$ and $C^*$ from (28) gives after some transformations

$$k\left[1 - (1 + \nu)\frac{T}{Eh^3} f_4\!\left(\frac{c}{R}\right)\right] = \frac{3PR^3}{2Gch^3}\left[1 - \left(\frac{c}{R}\right)^2\right]^2, \tag{33}$$

where

$$f_4(x) = \frac{6\left(\dfrac{1 - x^2}{2x}\right)^2 \ln\dfrac{1 + x}{1 - x}\left[\dfrac{1 + x^2}{2x}\ln\dfrac{1 + x}{1 - x} - 1\right]}{\left[1 - \left(\dfrac{1 - x^2}{2x}\ln\dfrac{1 + x}{1 - x}\right)^2\right]}. \tag{34}$$

Equation (33) shows that a positive moment $T$, that is a moment which tends to close an open ring sector, reduces the stiffness of the ring plate in so far as the effect of the axial force $P$ is concerned. In contrast to this a negative moment $T$ increases the transverse stiffness of the ring plate. For sufficiently large positive $T$ we have instability in the sense that we may have $k \neq 0$ when $P = 0$. It may, however, be that instability occurs in other modes of deformation for values of $T$ which are lower than that given by (33).

When $c/R$ is sufficiently small, we have the approximation

$$f_4(x) \sim 3\,\frac{1}{x}, \qquad x \lesssim \frac{1}{4}\,. \tag{34a}$$

In the range of applicability of (34a) we may further write

$$T = \frac{2}{3}\,\sigma_T\,c^2h, \tag{35}$$

where $\sigma_T \sim -\sigma_\theta(a) \sim \sigma_\theta(b)$. With this we have

$$\frac{T}{Eh^3}\,f_4\!\left(\frac{c}{R}\right) \sim 2\,\frac{\sigma_T}{E}\,\frac{cR}{h^2}, \qquad \frac{c}{R} \lesssim \frac{1}{4}\,. \tag{36}$$

The present section may be concluded by listing the form of the relation between $k$ and $P$ which holds when both the effect of $T$ and non-linearity is considered. We have then

$$k\left[1 + \frac{3(1+\nu)}{4}\,f_1\!\left(\frac{c}{R}\right)\!\left(\frac{k}{h}\right)^2 - (1+\nu)f_4\!\left(\frac{c}{R}\right)\frac{T}{Eh^3}\right] = \frac{3PR^3}{2Gch^3}\left[1 - \left(\frac{c}{R}\right)^2\right]^2. \tag{37}$$

We make the following further observation. Finite transverse deflections of the ring plate will be associated, in the absence of a moment $T$, with circumferential displacements parallel to the undeflected middle surface of the plate. Such displacements are also caused by the moment $T$. This means that if the conditions of load application are such that circumferential displacements are prevented $T$ will have a definite value, proportional to $k^2$ and the non-linear correction term in the transverse load-deflection relation for this form of load application will differ from the result (18) which holds when $T = 0$.

The problem of the determination of this modified correction term will be considered in the last section of this work, in conjunction with the problem of the ring plate with initial deflection.

**5. Pure twisting and bending of an initially deflected plate.** We now consider an initially deflected plate, or helicoidal shell, with middle-surface equation

$$W = K\theta. \tag{38}$$

We assume that $K$ is sufficiently small for the shell to be considered shallow. It seems that for practical purposes we may admit values of $K$ up to about $(R - c)/\pi$.

The differential equations for shallow shells which take the place of the flat-plate equations (1) and (2) have been given by Marguerre.[5] The necessary changes consist

[5]K. Marguerre, Proc. Vth Int. Congress Appl. Mech. Cambridge 1938, p. 98.

in replacing the operation $\mathfrak{D}(w)$ on the right of (2) by $\mathfrak{D}(W + w) - \mathfrak{D}(W)$ and in replacing $w$ on the right of (1) by $W + w$.

We may take as before

$$w = k\theta, \qquad F = F(r) \tag{9}$$

which reduces the two differential equations for $w$ and $F$ to the one equation

$$\nabla^2\nabla^2 F = Eh\,\frac{2Kk + k^2}{r^4}\,. \tag{39}$$

Equations (3), (4) and (5) which define stress resultants and couples remain unchanged while in equations (6) $w$ must be replaced by $W + w$.

The form of the boundary conditions (7), (8) and (27) remains unchanged and this means that we have now

$$\begin{aligned} F = \tfrac{1}{8}Eh(2Kk + k^2)[(\ln r)^2 + A \ln r + Br^2 + Cr^2 \ln r] \\ + T[A^* \ln r + B^* r^2 + C^* r^2 \ln r], \end{aligned} \tag{40}$$

where $A$, $B$ and $C$ and $A^*$, $B^*$ and $C^*$ are given by (16) and (31), respectively.

In determining the relation between force $P$ and deflection $k$ according to the fifth of Eqs. (8) it must be observed that $R_\theta$ of (12) is now changed to

$$R_\theta = -2(1 - \nu)D\,\frac{k}{r^3} + N_\theta\,\frac{K + k}{r}\,. \tag{41}$$

A comparison of (40) and (41) with the corresponding earlier results shows that Eq. (37) for $P$ as a function of $k$ and $T$ is changed into

$$\begin{aligned} k\left[1 + \frac{3(1+\nu)}{4} f_1\!\left(\frac{c}{R}\right)\frac{(k + 2K)(k + K)}{h^2} - (1+\nu)f_4\!\left(\frac{c}{R}\right)\frac{T}{Eh^3}\right] \\ - K\left[(1+\nu)f_4\!\left(\frac{c}{R}\right)\frac{T}{Eh^3}\right] = \frac{3PR^3}{2Gch^3}\left[1 - \left(\frac{c}{R}\right)^2\right]^2. \end{aligned} \tag{42}$$

As long as $k \ll K$ equation (42) reduces to the *linear* relation

$$k\left[1 + \frac{3(1+\nu)}{4} f_1\!\left(\frac{c}{R}\right)\frac{2K^2}{h^2}\right] - (1+\nu)f_4\!\left(\frac{c}{R}\right)\frac{KT}{Eh^3} = \frac{3PR^3}{2Gch^3}\left[1 - \left(\frac{c}{R}\right)^2\right]^2. \tag{43}$$

In view of the nature of the distribution of stresses $\sigma_\theta$ which give rise to the terms accounting for the initial deflection of the plate the term with $K^2$ can be obtained only by taking account of plate action and is consequently not incorporated in the classical theory of curved beams.

We note further that while for the initially undeflected plate the non-linear correction term varies as the cube of the deflection we have, for the initially deflected plate, correction terms varying both as the square and as the cube of the additional deflection caused by the forces $P$.

**6. Consideration of circumferential displacement.** We now consider in addition to the transverse displacement $w$ radial and circumferential displacement components

$u$ and $v$. Components of finite strain for the middle surface of the shallow shell are of the form

$$
\begin{aligned}
\epsilon_r &= \frac{\partial u}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial(W+w)}{\partial r}\right)^2 - \left(\frac{\partial W}{\partial r}\right)^2\right], \\
\epsilon_\theta &= \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2}\left[\left(\frac{\partial(W+w)}{r\,\partial \theta}\right)^2 - \left(\frac{\partial W}{r\,d\theta}\right)^2\right], \qquad (44) \\
\gamma_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \left[\frac{\partial(W+w)}{\partial r}\frac{\partial(W+w)}{r\,\partial\theta} - \frac{\partial W}{\partial r}\frac{\partial W}{r\,\partial\theta}\right].
\end{aligned}
$$

For the present problems we have $W = K\theta$, $w = k\theta$, $\partial u/\partial\theta = 0$ and $\gamma_{r\theta} = 0$. From this it follows that

$$v = \frac{\omega}{2\pi}\,\theta r, \qquad (45)$$

where $\omega$ is the relative angular displacement of the ends of one winding of the plate. The expressions for $\epsilon_r$ and $\epsilon_\theta$ reduce to

$$\epsilon_r = \frac{du}{dr}, \qquad \epsilon_\theta = \frac{u}{r} + \frac{\omega}{2\pi} + \frac{Kk + \frac{1}{2}k^2}{r^2}, \qquad (46)$$

leading to a compatibility relation of the form

$$\frac{dr\,\epsilon_\theta}{dr} - \epsilon_r = \frac{\omega}{2\pi} - \frac{Kk + \frac{1}{2}k^2}{r^2}. \qquad (47)$$

We introduce the relations

$$Eh\epsilon_\theta = N_\theta - \nu N_r, \qquad Eh\epsilon_r = N_r - \nu N_\theta \qquad (48)$$

and express $N_\theta$ and $N_r$ in terms of the stress function $F$ by means of (3), taking account of the fact that here $\partial F/\partial\theta = 0$. This leads to the result that

$$\frac{1}{Eh}\left[r\frac{d^3F}{dr^3} + \frac{d^2F}{dr^2} - \frac{1}{r}\frac{dF}{dr}\right] = \frac{\omega}{2\pi} - \frac{Kk + \frac{1}{2}k^2}{r^2} \qquad (49)$$

Into (49) we introduce $F$ from Eq. (40), and this gives us after some cancellations the relation

$$\frac{1}{2}(2Kk + k^2)C + \frac{4T}{Eh}C^* = \frac{\omega}{2\pi}, \qquad (50)$$

where $C$ and $C^*$ are defined by (16) and (31).

If we wish the axial force-displacement relation under the assumption of vanishing circumferential displacement we find from (50), with $\omega = 0$, as appropriate value of the couple $T$,

$$\frac{T}{Eh} = -\frac{(2Kk + k^2)C}{8C^*} = -\frac{2Kk + k^2}{8}\left[\frac{b^2 + a^2}{b^2 - a^2}\ln\frac{b}{a} - 1\right]\ln\frac{b}{a} \qquad (51)$$

Introduction of this value of $T$ into Eq. (42) gives after some transformations the following relation between $k$ and $P$,

$$k\left[1 + \frac{3(1+\nu)}{4} f_5\left(\frac{c}{R}\right) \frac{(K+k)(2K+k)}{h^2}\right] = \frac{3PR^3}{2Gch^3}\left[1 - \left(\frac{c}{R}\right)^2\right]^2, \tag{52}$$

where

$$f_5(x) = 1 - \left(\frac{1-x^2}{2x} \ln \frac{1+x}{1-x}\right)^2 \tag{53}$$

and

$$f_5(x) \sim \frac{4}{3} x^2, \qquad x \lesssim \frac{1}{2}, \qquad \lim_{x\to 1} f_5(x) = 1. \tag{54}$$

It may be seen that in the range $c/R < 1/4$ the effect of non-linearity and of initial deflection is much more pronounced when $\omega = 0$ than it is when $T = 0$, a factor $(4/15)\cdot(c/R)^4$ which occurs when $T = 0$ being replaced by a factor $(c/R)^2$ when $\omega = 0$. On the other hand, when $c/R$ is sufficiently near to unity the effect of non-linearity and of pretwist is the same for $T = 0$ and for $\omega = 0$.

TABLE I. Values of functions occurring in load displacement relations and stress ratio $\sigma_\theta/\tau_{r\theta}$.

| $x$ | $f_1$ | $f_2$ | $f_4$ | $f_5$ |
|---|---|---|---|---|
| 0 | 0 | 0 | $\infty$ | 0 |
| 0.1 | 0.000036 | 0.0159 | 29.08 | 0.0135 |
| 0.2 | 0.000574 | 0.0465 | 14.32 | 0.0530 |
| 0.3 | 0.00328 | 0.0836 | 8.97 | 0.1186 |
| 0.4 | 0.01024 | 0.1691 | 6.14 | 0.208 |
| 0.5 | 0.0265 | 0.255 | 4.31 | 0.321 |
| 0.6 | 0.0600 | 0.359 | 2.98 | 0.453 |
| 0.7 | 0.1248 | 0.483 | 1.94 | 0.600 |
| 0.8 | 0.2484 | 0.631 | 1.106 | 0.756 |
| 0.9 | 0.4923 | 0.809 | 0.427 | 0.903 |
| 1.0 | 1.0 | 1.0 | 0 | 1.0 |

# —NOTES—

## A SUFFICIENT CONDITION FOR AN INFINITE DISCRETE SPECTRUM*

By C. R. PUTNAM (*Purdue University*)

1. In the differential equation

$$x'' + f(t)x = 0, \tag{1}$$

let $f = f(t)$ denote a real-valued, continuous function on the half-line $0 \leqq t < \infty$. Both necessary and sufficient conditions in order that the equation (1) be oscillatory, so that every solution of (1) possesses an infinity of zeros on $0 \leqq t < \infty$ clustering at $+\infty$, are known; see, for instance, [10], [5]. The present note will deal primarily with the problem of obtaining a sufficient criterion in order that (1) be oscillatory, in the particular case that $f(t)$ satisfies the limit relation

$$f(t) \to 0, \qquad \text{as} \qquad t \to \infty. \tag{2}$$

The following will be shown:

(I) *If $f(t)$ satisfies* (2), *then the differential equation* (1) *is oscillatory whenever the inequality*

$$\limsup_{h \to +0} h\left(\limsup_{S \to \infty} S\left[\limsup_{T \to \infty} \int_S^T f(t)\,dt + \int_0^S f(t) \mid (t-\gamma)/(S-\gamma) \mid^{1+h} dt\right]\right) > \frac{1}{4} \tag{3}$$

*holds for every fixed number* $\gamma \geqq 0$.

Obviously, the inequality is satisfied in case the innermost "lim sup" is $+\infty$. (It should be pointed out here that if (2) holds and if $\lim_{T\to\infty}\int_0^T f(t)dt$ fails to exist either as a finite limit or as $-\infty$, then (1) is surely oscillatory; [3], p. 389. Cf. also [11] and [2].) It is noteworthy that the criterion furnished by (I) remains valid if the assumption (2) is replaced by certain other conditions; cf. the remark at the end of section 2.

The result (I) will have an implication concerning the discrete spectrum of the boundary value problem on $0 \leqq t < \infty$ ([7]) determined by the differential equation

$$x'' + (\lambda + f(t))x = 0 \tag{4}$$

and the boundary condition

$$x(0) \cos \alpha + x'(0) \sin \alpha = 0, \qquad 0 \leqq \alpha < \pi, \tag{5}$$

in the particular case that (2) holds. For every fixed $\alpha$, the relation (2) implies that the half-line $\lambda \geqq 0$ belongs to the spectrum; [2]. Moreover, only a discrete spectrum (isolated eigenvalues) can exist for $\lambda < 0$ and there exists a finite number of eigenvalues $\lambda < 0$, or an infinity of such eigenvalues clustering only at $\lambda = 0$, according as (1) is not or is oscillatory; [7], p. 252. Accordingly, (I) provides a sufficient condition that (4) and (5), in the case (2), determine a boundary value problem with an infinity of negative eigenvalues clustering at $\lambda = 0$. (It should be pointed out that (4) may have

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a non-trivial solution $x(\not\equiv 0)$ of class $L^2[0, \infty)$ for a positive value $\lambda$ even when (2) holds; cf. the constructions of [8], pp. 394-395 and [9], pp. 268-269.)

In section 3 below, a specific application of the criterion (I) to the spectral problem mentioned above will be made. The boundary value problem to be considered will be of the type arising in the quantum mechanical treatment of the two particle problem. In this case, there is a singularity in the coefficient function of the differential equation at the origin but the nature of the problem remains essentially identical with that considered in connection with (4) above; cf. [4], pp. 154, 163.

That the assertion of (I) can become false if the strict inequality $>$ of (3) is relaxed to $\geqq$ can easily be shown by an example. In fact, if $f(t) = Ct^{-2}$, where, say, $0 < t < \infty$, it is readily seen that (3) reduces to the inequality $C > \frac{1}{4}$. However, if $C = \frac{1}{4}$, so that $f(t)$ becomes $\frac{1}{4}t^{-2}$, then (1) possesses the non-oscillatory solution $x = t^{1/2}$, for $0 < t < \infty$. (Needless to say, the criterion (I) actually requires the continuity of the function $f(t)$ only for large values $t$ and the fundamental interval $0 \leqq t < \infty$ may be replaced by any half-line $T \leqq t < \infty$.)

2. The proof of (I) will depend upon an application of an oscillation criterion obtained in [5]. (A somewhat similar application was made in [6] in the case that $f(t)$ was periodic.) It was shown in [5] that (1) is oscillatory if and only if, for every $T \geqq 0$, there exists on $T \leqq t < \infty$ a continuous function $x = x(t)$ satisfying $x(T) = 0$ and possessing a piecewise continuous derivative $x'(t)$ such that each of the integrals

$$\int_T^\infty x^2(t)\,dt, \qquad \int_T^\infty x'^2(t)\,dt, \qquad \text{and} \qquad \int_T^\infty f(t)x^2(t)\,dt$$

is finite, while

$$\int_T^\infty (x'^2 - fx^2)\,dt < 0. \tag{6}$$

It will now be shown that relations (3) and (2) imply (6), and so, (I) will follow.

For every positive number $T$ and for every number $n > \frac{1}{2}$, consider numbers $T_2 < T_3 < T_4$ such that $T = T_1 < T_2$, and define the function $x = x(t)$ on $T_1 \leqq t < \infty$ as follows:

$$x(t) = \begin{cases} (t - T_1)^n, & \text{for} \quad T_1 \leqq t \leqq T_2\,; \\ (T_2 - T_1)^n, & \text{for} \quad T_2 \leqq t \leqq T_3\,; \\ (T_2 - T_1)^n(T_4 - T_3)^{-1}(T_4 - t), & \text{for} \quad T_3 \leqq t \leqq T_4\,; \\ 0, & \text{for} \quad T_4 \leqq t < \infty. \end{cases}$$

If $T_2 - T_1 = A$ and $T_4 - T_3 = B$, direct calculation readily shows that the requirement (6) reduces to

$$A \int_{T_2}^{T_3} f\,dt + A^{1-2n} \int_{T_1}^{T_2} f(t - T_1)^{2n}\,dt$$
$$+ AB^{-2} \int_{T_3}^{T_4} f(T_4 - t)^2\,dt > n^2(2n - 1)^{-1} + AB^{-1}.$$

Suppose that $T_2$ (hence $A$) is determined and then, for an arbitrary positive number $\epsilon$, choose $B$ so as to satisfy $AB^{-1} < \epsilon$. Next choose $T_3$ (hence $T_4$) so large that

$$\left| AB^{-2} \int_{T_3}^{T_4} f(T_4 - t)^2 \, dt \right| < \epsilon. \tag{7}$$

That this can be done is clear from (2) and from the fact that $| T_4 - t |^2 B^{-2} \leqq 1$ when $T_3 \leqq t \leqq T_4$ $(= T_3 + B)$. It now follows that (6) is surely satisfied if

$$A \int_{T_2}^{T_3} f \, dt + A^{1-2n} \int_{T_1}^{T_2} f(t - T_1)^{2n} \, dt > n^2(2n - 1)^{-1} + 2\epsilon. \tag{8}$$

Actually it will be shown that (8) holds for (certain) large values $A$, $T_k$ (that is, for certain $A$, $T_k \to \infty$) as a consequence of (3).

Let $L(S)$ be defined by

$$L(S) = \limsup_{T \to \infty} \int_S^T f(t) \, dt, \tag{9}$$

and for a fixed value $T_2$, choose $T_3$ so large that

$$\int_{T_2}^{T_3} f \, dt > L(T_2) - T_2^{-2}.$$

It is clear that for a fixed value $T_1$, $AT_2^{-1} \to 1$ as $T_2 \to \infty$, and so, for certain large $A$, $T_k$, the relation (8) will hold if $T_1$ is fixed and

$$T_2 L(T_2) + T_2 \int_{T_1}^{T_2} f[(t - T_1)/A]^{2n} \, dt > n^2(2n - 1)^{-1} + 2\epsilon$$

holds for certain large $T_2$. However, this last relation will certainly hold for certain large $T_2$ ($T_1$ fixed) if

$$\limsup_{S \to \infty} \left[ SL(S) + S \int_0^S f \, | (t - T_1)/(S - T_1) |^{2n} \, dt \right] > n^2(2n - 1)^{-1} + 2\epsilon. \tag{10}$$

Since $\epsilon$ can be chosen arbitrarily small, the $2\epsilon$ appearing on the right side of (10) may be deleted. If now $h = 2n - 1$, one sees that (6) holds for every $T$ $(= T_1 = \gamma)$ if the relation

$$h \limsup_{S \to \infty} S \left[ L(S) + \int_0^S f(t) \, | (t - \gamma)/(S - \gamma) |^{1+h} \, dt \right] > (1 + h)^2/4 \tag{11}$$

holds for some $h > 0$ (where possibly $h = h(\gamma)$) and $\gamma$ is an arbitrary non-negative number. However, relations (3) and (9) clearly imply that (11), for every $\gamma \geqq 0$, is valid for some positive $h$ and thus the assertion (I) is proved.

*Remark.* The assumption (2) was used in order to obtain (7). It is clear though from the proof given above that (3) will imply (6) if

$$\left| AB^{-2} \int_{T_3}^{T_3+B} f^-(t)(T_3 + B - t)^2 \, dt \right| < \epsilon, \qquad f^- = \begin{cases} f, & f \leq 0 \\ 0, & f > 0 \end{cases}$$

can be obtained for $A$, $B$ fixed and for (certain) large $T_s$. Thus, the criterion furnished by (I) will be valid if, for instance, the restriction (2) is replaced by any one of the three assumptions (i) $f(t) \geqq 0$, or even (ii) $f^-(t) \to 0$ as $t \to \infty$, or

$$\text{(iii)} \qquad \left| \int_0^\infty f^-(t)\, dt \right| < \infty.$$

3. In order to obtain an application of (I), consider the radial portion of the separated form of the quantum mechanical wave equation of the two particle problem, namely,

$$R'' + c(\lambda - V(r) - l(l+1)/cr^2)R = 0;$$

cf. [4], p. 150. Here, $c$, $\lambda$ and $l$ are constants and the prime denotes differentiation with respect to $r$. It will be assumed that $V(r) \to 0$ as $r \to \infty$; cf. [4], p. 152. It is clear from an earlier remark that, as far as concerns the application of (I), only the continuity of the function $V(r)$ for large $r$, say for $1 \leqq r < \infty$, is needed. (As is customary, it will be assumed that the singularity of $V(r)$ at $r = 0$ is of a suitably restricted type; cf., e.g., [4], pp. 152, 163.) It follows from (I) and the calculation of section 1 that the equation

$$R'' + c(-V(r) - l(l+1)/cr^2)R = 0 \tag{12}$$

is oscillatory if, for all $\gamma \geqq 0$,

$$\limsup_{h\to+0} h\left(\limsup_{S\to\infty} S\left[\limsup_{T\to\infty} \int_S^T - cV(r)\, dr + \int_1^S - cV(r)\, |(r-\gamma)/(S-\gamma)|^{1+h}\, dr\right]\right)$$
$$> \tfrac{1}{4} + l(l+1). \tag{13}$$

Thus, if the last inequality is satisfied, there exists an infinity of negative energy levels clustering at $\lambda = 0$. In the case of the hydrogen atom, $V(r) = kr^{-2}$, where $k < 0$, so that the bracketed portion of the left side of (13) is $+\infty$ and hence (12) is oscillatory for all values $l$; cf., e.g., [4], pp. 157 ff. In many cases however, the potential $V(r)$ appears to be unknown (cf. [4], p. 156 and [1], pp. 30 ff.) and the relation (13) offers a property of $V(r)$ guaranteeing the existence of an infinite (negative) discrete spectrum.

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# THE TORSION AND STRETCHING OF SPIRAL RODS (II)*

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The torsion and stretching problems of spiral rods were discussed in a preceding paper.[1] There, the equations of equilibrium were expressed in terms of displacements that were independent of the position of the section perpendicular to the axis of a spiral rod. The differential equations for the displacements were integrated for the particular case where the helix angle was small, and the corresponding displacements and stresses were obtained. In the calculations, however, the displacements were preliminarily assumed in special forms, and consequently solution was valid for some special problems. In the previous paper, the displacements for the stretching problem were assumed in forms that reduce to those for a uniform tension in the limit case when the helix angle approaches zero. But when a spiral rod with axis that does not pass the centroid of the cross section, is pulled axially, the displacements must be in forms that reduce to those for a uniform tension combined with a uniform bending moment in the limit case when the helix angle approaches zero. Hence, the validity of the previous solution was restricted to the problem for a spiral rod with axis through the centroid of the cross section.

As in the preceding paper, we take the axis of helix as the axis of $z$, and denote the displacements in $x'$, $y'$, $z$ directions by $u'$, $v'$ and $w$ respectively, in which $x'$ and $y'$ are the axes perpendicular to each other and fixed to a section of the rod perpendicular to $z$. We take for the displacements the expressions

$$\left.\begin{aligned} u' &= u_1 - \gamma x' - \frac{\gamma'}{2}(x'^2 - y'^2) - \alpha y'z + \frac{\beta'}{k^2}(1 - \cos kz - kz\sin kz), \\ v' &= v_1 - \gamma y' - \gamma' x'y' + \alpha x'z - \frac{\beta'}{k^2}(\sin kz - kz\cos kz), \\ w &= w_1 + \frac{\beta'}{k}(x'\sin kz - y'\cos kz + y') + \beta z, \end{aligned}\right\} \tag{1}$$

where $u_1$, $v_1$, $w_1$ are the functions of $x'$, $y'$ and are independent of $z$, $k$ is a constant which specifies the helix angle, $\alpha$, $\beta$, $\beta'$ are arbitrary constants, and

$$\gamma = \frac{1}{2}(1 - p)\beta, \qquad \gamma' = \frac{1}{2}(1 - p)\beta', \qquad p = \frac{\mu}{\lambda + \mu}.$$

From (1) we have the cubical dilatation

$$\Delta = \frac{\partial u_1}{\partial x'} + \frac{\partial v_1}{\partial y'} - kD_2(w_1) + p\beta' x' + p\beta,$$

where

$$D_2 = y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}.$$

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[1]H. Ōkubo, Q. Appl. Math. 9, 263-272 (1951).

The equations of equilibrium for this case can be expressed in the forms

$$\left.\begin{aligned} \frac{\partial\Delta}{\partial x'} + p\{\nabla_1^2 u_1 + k^2 D_1(u_1) - 2k^2 D_2(v_1) + \frac{1}{2}k^2\gamma'(x'^2 - y'^2) - \beta'\} &= 0,\\ \frac{\partial\Delta}{\partial y'} + p\{\nabla_1^2 v_1 + k^2 D_1(v_1) + 2k^2 D_2(u_1) + k^2\gamma' x'y'\} &= 0,\\ kD_2(\Delta) - p\{\nabla_1^2 w_1 + k^2 D_1(w_1) + k^2 w_1 - k\beta' y'\} &= 0, \end{aligned}\right\} \tag{2}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}, \qquad \text{and} \qquad D_1 + 1 = \left(y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}\right)\left(y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}\right).$$

The differential equations (2) are independent of $z$. Solving the simultaneous equations for $u_1$, $v_1$ and $w_1$, we can find the displacements from (1). The displacements for a straight rod are readily obtained from (1), by taking the limit case when $k$ approaches zero; thus

$$\left.\begin{aligned} u &= -\gamma x - \alpha yz - \frac{\gamma'}{2}(x^2 - y^2) - \frac{\beta'}{2}z^2,\\ v &= -\gamma y + \alpha xz - \gamma' xy,\\ w &= w_1 + \beta z + \beta' xz, \end{aligned}\right\} \tag{3}$$

where $u_1$, $v_1$ are assumed to vanish when $k$ approaches zero.

Assume that $w_1$ and $\alpha$ in (3) also vanish when $k$ approaches zero; the corresponding stresses become

$$X_x = Y_y = X_y = X_z = Y_z = 0, \qquad Z_z = \mu(3 - p)(\beta + \beta' x). \tag{4}$$

This is the solution for a straight rod submitted to simple tension combined with a uniform bending moment. If we assume that $\beta$ and $\beta'$, instead of $w_1$ and $\alpha$, vanish when $k$ approaches zero, the corresponding stresses become

$$X_x = Y_y = Z_z = X_y = 0, \qquad X_z = \mu\left(\frac{\partial w_1}{\partial x} - \alpha y\right), \qquad Y_z = \mu\left(\frac{\partial w_1}{\partial y} + \alpha x\right). \tag{5}$$

This is the solution for the torsion problem of a straight rod.

Consider now a spiral rod of small $k$, pulled by a pair of axial forces. Assuming that $\alpha$, $w_1$ are small quantities of the order $k$ and $u_1$, $v_1$ are of the order $k^2$, and neglecting the small quanties of the higher order, the equations of equilibrium (2) can be written as follows:

$$\left.\begin{aligned} \frac{\partial\Delta}{\partial x'} + p\nabla_1^2 u_1 + \frac{1}{2}pk^2\gamma'(x'^2 - y'^2) - p\beta' &= 0,\\ \frac{\partial\Delta}{\partial y'} + p\nabla_1^2 v_1 + pk^2\gamma' x'y' &= 0,\\ \nabla_1^2 w_1 - 2k\beta' y' &= 0. \end{aligned}\right\} \tag{6}$$

Let us take for $w_1$ the expression

$$w_1 = i(f_3 - \bar{f}_3) + \tfrac{1}{3}k\beta' y'^3, \tag{7}$$

where $f_3$ is an arbitrary function, $\bar{f}_3$ is a function conjugate with $f_3$, and

$$f_3 = f_3(\zeta), \qquad \bar{f}_3 = \bar{f}_3(\bar{\zeta}), \qquad \zeta = x' + iy', \qquad \bar{\zeta} = x' - iy'.$$

The displacement $w_1$ satisfies the third Eq. (6). Substituting this expression of $w_1$ into the first and second equations (6), the equations for $u_1$ and $v_1$ become

$$\left.\begin{aligned}
&(1 + p)\frac{\partial^2 u_1}{\partial x'^2} + \frac{\partial^2 v_1}{\partial x'\,\partial y'} + p\frac{\partial^2 u_1}{\partial y'^2}\\
&\qquad = k(f_3' + \bar{f}_3' + \zeta f_3'' + \bar{\zeta}\bar{f}_3'') - \frac{1}{2}pk^2\gamma' x'^2 - \frac{1}{4}(4 - p + p^2)k^2\beta' y'^2,\\
&p\frac{\partial^2 v_1}{\partial x'^2} + \frac{\partial^2 u_1}{\partial x'\partial y'} + (1 + p)\frac{\partial^2 v_1}{\partial y'^2}\\
&\qquad = ik(f_3' - \bar{f}_3' + \zeta f_3'' - \bar{\zeta}\bar{f}_3'') - \frac{1}{2}(4 + p - p^2)k^2\beta' x'y'.
\end{aligned}\right\} \tag{8}$$

Integrating the differential equations (8), we find

$$\left.\begin{aligned}
u_1 &= f_1 + \bar{f}_1 + x'(f_2 + \bar{f}_2) + k\int f_3''\zeta\,d\zeta + k\int \bar{f}_3''\bar{\zeta}\,d\bar{\zeta}\\
&\quad + \frac{k^2\beta'}{48}\{p(3 - p)x'^4 - 6(4 + p - p^2)x'^2y'^2 + (6 - p - p^2)y'^4\},\\
v_1 &= i(f_1 - \bar{f}_1) + ix'(f_2 - \bar{f}_2) + i(1 + 2p)\left(\int f_2\,d\zeta - \int \bar{f}_2\,d\bar{\zeta}\right),
\end{aligned}\right\} \tag{9}$$

where $f_1$, $f_2$ are arbitrary functions of $\zeta$. The corresponding stresses become

$$\left.\begin{aligned}
X'_{x'} &= 2\mu\Big\{f_1' + \bar{f}_1' + x'(f_2' + \bar{f}_2') + p(f_2 + \bar{f}_2) + k(\zeta f_3' + \bar{\zeta}\bar{f}_3')\\
&\qquad + \frac{1}{24}(3 - p)(1 + p)k^2\beta' x'^3 - \frac{1}{8}(9 - p^2)k^2\beta' x'y'^2\Big\},\\
Y'_{y'} &= -2\mu\Big\{f_1' + \bar{f}_1' + x'(f_2' + \bar{f}_2') + (2 + p)(f_2 + \bar{f}_2)\\
&\qquad - \frac{1}{24}(3 - p)(1 - p)k^2\beta' x'^3 + \frac{1}{8}(1 - p)^2k^2\beta' x'y'^2\Big\},\\
Z_z &= -2\mu\Big\{(1 - p)(f_2 + \bar{f}_2) + k(\zeta f_3' + \bar{\zeta}\bar{f}_3') - \frac{1}{24}(1 - p)(3 - p)k^2\beta' x'^3\\
&\qquad - \frac{1}{8}(7 + 2p - p^2)k^2\beta' x'y'^2 - \frac{1}{2}(3 - p)(\beta + \beta' x')\Big\};\\
X'_{y'} &= 2i\mu\Big\{f_1' - \bar{f}_1' + x'(f_2' - \bar{f}_2') + (1 + p)(f_2 - \bar{f}_2) + \frac{k}{2}(\zeta f_3' - \bar{\zeta}\bar{f}_3')\\
&\qquad + \frac{i}{8}(4 + p - p^2)k^2\beta' x'^2y' - \frac{i}{24}(6 - p - p^2)k^2\beta' y'^3\Big\},\\
X'_z &= \mu\{i(f_3' - \bar{f}_3') - \alpha y' + k\gamma' x'y'\},\\
Y'_z &= -\mu\Big\{f_3' + \bar{f}_3' - \alpha x' - k\beta' y'^2 + \frac{k\gamma'}{2}(x'^2 - y'^2)\Big\}.
\end{aligned}\right\} \tag{10}$$

Take for the bounding curve of the section, the expression

$$F(x', y') = 0. \tag{11}$$

The conditions for the lateral surface of the rod to be free from external forces are

$$\left.\begin{aligned} X'_{x'}\frac{\partial F}{\partial x'} + X'_{y'}\frac{\partial F}{\partial y'} - kD_2(F)X'_z &= 0, \\ X'_{y'}\frac{\partial F}{\partial x'} + Y'_{y'}\frac{\partial F}{\partial y'} - kD_2(F)Y'_z &= 0, \\ X'_z\frac{\partial F}{\partial x'} + Y'_z\frac{\partial F}{\partial y'} - \mu k(3 - p)(\beta + \beta' x')D_2(F) &= 0. \end{aligned}\right\} \tag{12}$$

Consider a spiral rod stretched by a pair of axial forces $P$, and imagine a small portion of the rod cut by two parallel planes perpendicular to the axis of rod, as shown in Fig. 1.

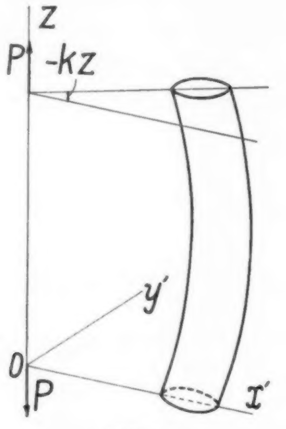


FIG. 1.

The equilibrium condition of the surface tractions for this portion is

$$\left.\begin{aligned} \iint Z_z\, dx'\, dy' &= P, \\ \iint Z_z x'\, dx'\, dy' &= 0, \\ \iint (y'X'_z - x'Y'_z)\, dx'\, dy' &= 0, \end{aligned}\right\} \tag{13}$$

where the integrals are taken over the cross section.

The arbitrary functions $f_1$, $f_2$ and $f_3$ are determined so as to satisfy the boundary condition (12), and the constants $\alpha$, $\beta$ and $\beta'$ are obtained from the condition (13).

Substituting the expressions of $X'_z$, $Y'_z$ in (10) into the third Eq. (12), it becomes

$$\frac{d}{ds}(f_3 + \bar{f}_3) - \frac{1}{2}\{\alpha + (3-p)k\beta\}\frac{d}{ds}(\zeta\bar{\zeta}) - \frac{1}{12}(11-3p)k\beta'\frac{dx'^3}{ds} - \frac{1}{4}(5-p)k\beta'\frac{d}{ds}x'y'^2 = 0,$$

where $ds$ is the element of arc of the bounding curve of the cross section. It follows that the equation

$$f_3 + \bar{f}_3 = \frac{1}{2}\{\alpha + (3-p)k\beta\}(x'^2 + y'^2) + \frac{1}{12}(11-3p)k\beta'x'^3 + \frac{1}{4}(5-p)k\beta'x'y'^2 + \text{const.}, \tag{14}$$

holds on the bounding curve, from which $f_3$ is determined. For the convenience of further calculations, we shall rewrite the stresses $X'_{x'}$, $Y'_{y'}$ and $X'_{y'}$ in (11) in the expressions as

$$\left.\begin{aligned} X'_{x'} &= \frac{\partial^2 \chi}{\partial y'^2} + 2\mu k(\zeta f'_3 + \bar{\zeta}\bar{f}'_3) + \frac{\mu}{12}(3-p)(1+p)k^2\beta'x'^3 - \frac{\mu}{4}(9-p^2)k^2\beta'x'y'^2, \\ &\vdots \end{aligned}\right\} \tag{15}$$

where

$$\chi = -\varphi_1 - \bar{\varphi}_1 - x'(\varphi_2 + \bar{\varphi}_2), \qquad \varphi_1'' = 2\mu(f_1' + pf_2), \qquad \varphi_2' = 2\mu f_2 .$$

Substituting these expressions for stresses into the first and second Eqs. (12), by virtue of (14), we obtain

$$\left.\begin{aligned} &\frac{d}{ds}\left(\frac{\partial \chi}{\partial x'} + i\frac{\partial \chi}{\partial y'}\right) + \mu k\frac{d}{ds}[F_3(\zeta) - \bar{F}_3(\bar{\zeta})] + \mu k^2\beta'\frac{d\Phi}{ds} \\ &\qquad - \frac{1}{2}(3-p)\mu k^2\beta'\zeta\frac{d}{ds}(\zeta\bar{\zeta}) - \frac{i}{4}(3-p)\mu k^2\beta'x'^2y'\frac{d}{ds}(\zeta + 3\bar{\zeta}) = 0, \\ &\frac{d}{ds}\left(\frac{\partial \chi}{\partial x'} - i\frac{\partial \chi}{\partial y'}\right) - \mu k\frac{d}{ds}[F_3(\zeta) - \bar{F}_3(\bar{\zeta})] + \mu k^2\beta'\frac{d\bar{\Phi}}{ds} \\ &\qquad - \frac{1}{2}(3-p)\mu k^2\beta'\bar{\zeta}\frac{d}{ds}(\zeta\bar{\zeta}) + \frac{i}{4}(3-p)\mu k^2\beta'x'^2y'\frac{d}{ds}(3\zeta + \bar{\zeta}) = 0, \end{aligned}\right\} \tag{16}$$

where

$$F_3' = 2\zeta f_3' ,$$

$$\Phi = -\frac{1}{48}[(11 + p)(3 - p)x'^4 + 6(1 - p)^2x'^2y'^2 - (9 - 2p + p^2)y'^4]$$

$$+ \frac{i}{12}(3 - p)[(1 + p)x'^2 - (7 + p)y'^2]x'y'.$$

From (16), $f_1$ and $f_2$ are obtained.

As an example of the procedure, consider an elliptic spiral rod whose cross section is

$$(x' - c)^2/a^2 + y'^2/b^2 = 1, \tag{17}$$

as shown in Fig. 2.

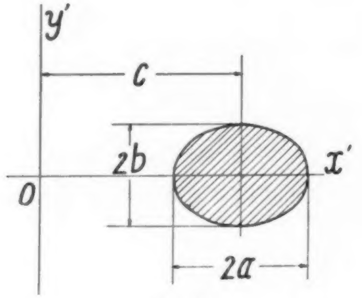


Fig. 2.

Transform now the elliptic section in the $\zeta$-plane into a unit circle in the $t$-plane by the equation

$$\zeta = c + a't + b't^{-1}, \tag{18}$$

where $a' + b' = a$, $a' - b' = b$.

Take for the function $f_3$, the expression

$$f_3 = c_1(t + st^{-1}) + c_2(t^2 + s^2t^{-2}) + c_3(t^3 + s^3t^{-3}), \tag{19}$$

where $S = b'/a'$. The unknown coefficients $c_1$, $c_2$ and $c_3$ are readily obtained from the boundary condition (14).

Remembering the condition that $\varphi_1$ and $\varphi_2$ are analytic at any point of the section in the $t$-plane, we take for the functions the expressions

$$\varphi_1' = \sum_{n=1}^{4} A_n(t^n + s^nt^{-n}), \qquad \varphi_2 = \sum_{n=1}^{4} B_n(t^n + s^nt^{-n}), \tag{20}$$

where $A_n$ and $B_n$ are real constants. These unknown constants are obtained from the boundary condition (16), and the other unknown constants $\alpha$, $\beta$ and $\beta'$ are finally obtained from the conditions (13).

In the case of stretching, the predominating stress is the normal stress $Z_z$ and the shearing stresses $X_z'$ and $Y_z'$ follow it, but the latter are smaller quantities of the order $k$. The other stresses are of the order $k^2$ and are very small quantities when $k$ is small. When the section of the spiral rod is a circle of radius unity, then $a = b = 1$, and it

follows $a' = 1$, $b' = 0$, $s = 0$. The main stress $Z_z$ for the circular section, referred to the polar coordinates with the pole at the center of the circle, is

$$\begin{aligned} Z_z = {} & \mu(3 - p)(\beta + \beta' c) - 2(1 - p)B_1 - 4\mu k c_1 c \\ & + [\mu\beta'(3 - p) - 4(1 - p)B_2 - 4\mu k(2c_2 c + c_1)]r \cos\theta \\ & - 2[3(1 - p)B_3 + 2\mu k(3c_3 c + 2c_2)]r^2 \cos 2\theta - 4[2(1 - p)B_4 + 3\mu k c_3]r^3 \cos 3\theta \\ & + \frac{1}{12}\mu k^2\beta'(1 - p)(3 - p)(c + r\cos\theta)^3 \\ & + \frac{1}{4}\mu k^2\beta'(7 + 2p - p^2)(c + r\cos\theta)r^2 \sin^2\theta. \end{aligned} \tag{21}$$

From the third Eq. (13), we have

$$\alpha = \tfrac{1}{2}(1 - p)k\beta' c. \tag{22}$$

Hence, a twist almost proportional to $k$ arises when a spiral rod with a circular section


FIG. 3. The shearing stress along the bounding circle, when $k = 0.25$.

is pulled axially.[2] From the remaining Eqs. (13), the other unknown constants $\beta$ and $\beta'$ are obtained.

If $\tau$ be the shearing stress along the bounding circle of the section, it becomes

$$\} = -\mu\{[2c_1 - \tfrac{1}{4}k\beta' c^2(1 - p) - \tfrac{1}{4}pk\beta'] \cos \theta + 4c_2 \cos 2\theta + (6c_3 + \tfrac{1}{4}k\beta') \cos 3\theta\}. \quad (23)$$

The shearing stress along the periphery has been calculated from (23) for various values of $c$, assuming the Poisson's ratio and $k$ to be 0.3 and 0.25, respectively, and is shown in Fig. 3. As is seen from the figure, the shearing stress becomes large at both ends of two diameters parallel to the coordinate axes $(x', y')$, and attains its maximum value at the outer end of the diameter on the $x'$-axis. The distribution of the normal tension $Z_z$ on the axis of $x'$, obtained from (21), is given in Fig. 4. For the sake of comparison, the

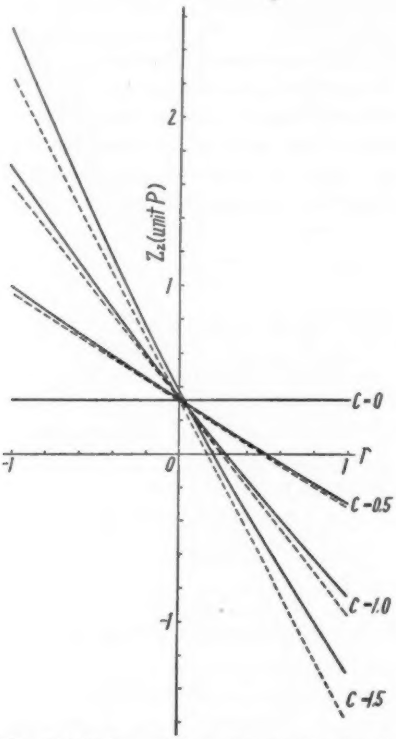


FIG. 4. The distribution of $Z_z$ on the $x'$-axis, when $k = 0.25$.

corresponding distribution of $Z_z$ for a straight rod is also shown by dotted lines in the same figure.

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[2] $\beta'$ is a function of $k$ and $c$, but it remains almost constant for the variation of $k$, when $k$ is small.

## ON THE GAPS IN THE SPECTRUM OF THE HILL EQUATION*

By C. R. PUTNAM (*Purdue University*)

**1.** Let $f = f(t)$ be a real-valued, continuous, periodic function of period 1, so that

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n \exp(2\pi \text{ int}), \qquad (c_{-n} = \bar{c}_n), \tag{1}$$

and consider the Hill equation

$$x'' + (\lambda + f(t))x = 0, \qquad (\lambda \text{ real}; \ ' = d/dt). \tag{2}$$

It is known that (if $f \not\equiv 0$) there exists a sequence of closed intervals $I_k$: $\lambda_k \leq \lambda \leq \lambda^k$ (region of stability), where $\lambda_k < \lambda^k < \lambda_{k+1}$ and $k = 1, 2, \cdots$, with the property that (2) has some solution $x \not\equiv 0$ which is bounded on $-\infty < t < \infty$ if and only if $\lambda$ belongs to the closed set $S = \sum I_k$; cf. [7], p. 14. The complementary set of $S$ consists of a half-line $-\infty < \lambda < \lambda_1$ and the sequence of open intervals $J_k : \lambda^k < \lambda < \lambda_{k+1}$, $k = 1, 2, \cdots$. In several recent papers, various lower bounds for the value $\lambda_1$, the least point of the set $S$, in terms of the Fourier coefficients $c_n$ of $f(t)$, have been obtained; [11], [5], [3]. The present note will be devoted to the problem of obtaining estimates (upper bounds) of the lengths $\lambda_{k+1} - \lambda^k$ of the "gaps" $J_k$ of the set $S$ in terms of these Fourier coefficients.

It follows from [4], p. 613, that the length of every gap $J_k$ is surely not greater than

$$\limsup_{t\to\infty} f(t) - \liminf_{t\to\infty} f(t) \leq 4 \sum_{n=1}^{\infty} | c_n |. \tag{3}$$

In addition, asymptotic estimates, as $\lambda^k \to \infty$, for these gaps are known; [2]. In fact, since $f(t)$ is uniformly continuous on $0 \leq t < \infty$, the lengths $\lambda_{k+1} - \lambda^k$ of the intervals $J_k$ tend to zero as $\lambda_{k+1} \to \infty$; *loc. cit.*, p. 850. Furthermore, additional regularity conditions on $f(t)$ result in more refined estimates. It should be pointed out here that the investigations of [2] related to singular boundary value problems ([8]) on the half-line $0 \leq t < \infty$ determined by (2) and a linear, homogeneous boundary condition at $t = 0$, and were not confined to the special case that $f(t)$ be periodic.

Let $m(\lambda)$, for $-\infty < \lambda < \infty$, be defined to be the distance from $\lambda$ to the set $S$ considered above, so that

$$m(\lambda) = \text{g.l.b.} \ | \lambda - \mu |, \qquad \mu \text{ in } S. \tag{4}$$

It will be shown in section 2 below that $m(\lambda)$ satisfies the inequality

$$m^2(\lambda) \leq 2 \sum_{n=1}^{\infty} | c_n |^2, \qquad \text{provided} \qquad \lambda \geq -c_0 . \tag{5}$$

As a consequence of (4) and (5), one readily sees that the lengths $\lambda_{k+1} - \lambda^k$ of the gaps $J_k$ satisfy

$$\lambda_{k+1} - \lambda^k \leq 2\left(2 \sum_{n=1}^{\infty} | c_n |^2\right)^{1/2}, \qquad \text{provided} \qquad \frac{1}{2}(\lambda_{k+1} + \lambda^k) \geq -c_0 . \tag{6}$$

It will remain undecided whether (6) actually must hold for *all* gaps $J_k$, so that the first inequality of (6) would hold without the proviso of the second inequality. In any

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case, it is readily seen that the estimate of (6), when it applies, is an improvement over that of (3), namely $4 \sum_{n=1}^{\infty} | c_n |$.

In this connection, it should be pointed out that Kato [3], by an adaptation of a relation used by Wintner [11], has obtained the inequality

$$\lambda_1 \geq -c_0 - \left(\frac{1}{8}\right) \sum_{n=1}^{\infty} | c_n |^2,$$

for the least point $\lambda_1$ of the set $S$. (Wintner had previously shown that $\lambda_1 \geq -c_0 - 2 \cdot \sum_{n=1}^{\infty} | c_n |^2$.) Consequently, it is easily seen that the first inequality of (6) is surely valid for *all* gaps $J_k$ if, for instance, the inequality

$$\left(\frac{1}{8}\right) \sum_{n=1}^{\infty} | c_n |^2 \leq \left(2 \sum_{n=1}^{\infty} | c_n |^2\right)^{1/2}$$

holds. (If one normalizes $f$ so that its mean value is zero, hence $c_0 = 0$, this last inequality is equivalent to $\int_0^1 f^2 \, dt \leq 256$).

Before proceeding to the proof of (5), it can be noted that the first inequality of (5) surely becomes false if the restriction $\lambda \geq -c_0$ is dropped. In fact, if $f(t) \equiv c_0$, so that (2) becomes the differential equation of the harmonic oscillator, then $\sum_{n=1}^{\infty} | c_n |^2 = 0$, and (5) yields the known result that $m(\lambda) \equiv 0$ for $\lambda \geq -c_0$. However, $m(\lambda) > 0$ for $\lambda < -c_0$, since $S$ is the half-line $-c_0 \leq \lambda < \infty$.

2. The proof of (5) will depend upon certain results obtained in [6]. Let $g_1(t), g_2(t), \cdots$, denote a sequence of functions possessing continuous second derivatives on $0 \leq t < \infty$, satisfying

$$g_n(0) = g_n'(0) = 0, \tag{7}$$

and such that $g_n(t) \to 0$ uniformly on every finite $t$-interval $[0, T]$. Then, if $g_n$ and $L(g_n)$ (where $L(x) \equiv x'' + fx$) are of class $L^2 [0, \infty)$, the inequality

$$m^2(\lambda) \liminf_{n\to\infty} \int_0^{\infty} g_n^2 \, dt \leq \liminf_{n\to\infty} \int_0^{\infty} (L(g_n) + \lambda g_n)^2 \, dt \tag{8}$$

holds. This follows readily by a method analogous to that given in [6], p. 580. (It is to be noted that the set $S$ considered above is identical with the invariant spectrum (Weyl [8], p. 251) associated with the differential equation (2); [9], [1]. Moreover, the investigations of [6] related to the Weyl theory of singular boundary value problems, alluded to in section 1.)

Next, let $\mu > 0$, and let $g_n = y_n h$, where $h = \sin (\mu^{\frac{1}{2}} t)$ or $h = \cos (\mu^{\frac{1}{2}} t)$, and the $y_n = y_n(t)$ are functions possessing continuous second derivatives on $0 \leq t < \infty$. In addition, suppose that $y_n(0) = y_n'(0) = 0$, so that (7) certainly holds, and that $y_n$ and $L(y_n)$ belong to $L^2(0, \infty)$. Finally, suppose that the $y_n$ are such that the "lim inf" appearing on the left side of the inequality (8) can be replaced by "lim" for both $h = \sin (\mu^{\frac{1}{2}} t)$ and $h = \cos (\mu^{\frac{1}{2}} t)$.

It follows from (8) that

$$m^2(\lambda) \lim_{n\to\infty} \int_0^{\infty} y_n^2 h^2 \, dt \leq \liminf_{n\to\infty} \int_0^{\infty} ([y_n'' + (\lambda - \mu + f) y_n] h + 2 y_n' h')^2 \, dt.$$

If now the $y_n$ satisfy

$$\int_0^{\infty} y_n'^2 \, dt \to 0, \qquad \int_0^{\infty} y_n''^2 \, dt \to 0, \qquad (n \to \infty),$$

it is seen that

$$m^2(\lambda) \lim_{n\to\infty} \int_0^\infty y_n^2 h^2 \, dt \le \liminf_{n\to\infty} \int_0^\infty (\lambda - \mu + f)^2 y_n^2 h^2 \, dt. \tag{9}$$

Since (9) holds for both functions $h$, addition of the two corresponding inequality relations yields, in view of the fact that $\liminf A + \liminf B \le \liminf (A + B)$, the inequality

$$m^2(\lambda) \lim_{n\to\infty} \int_0^\infty y_n^2 \, dt \le \liminf_{n\to\infty} \int_0^\infty (\lambda - \mu + f)^2 y_n^2 \, dt. \tag{10}$$

Let $T > 0$ and define the function $Y_T(t)$ on $0 \le t < \infty$ so that the graph of $Y_T(t)$ on $0 \le t \le T$ consists of three line segments joining, in order, the four points (0, 0), $(1, T^{-\frac{1}{2}})$, $(T - 1, T^{-\frac{1}{2}})$, and $(T, 0)$. On $T < t < \infty$, let $Y_T(t) \equiv 0$. It is clear that the corners of this function can be smoothed out so as to obtain a function $y_T(t)$ satisfying the conditions imposed upon the $y_n$ above. Furthermore, it is clear that if $y_n = y_{T_n}$, where $T = T_n \to \infty$ as $n \to \infty$, one can arrange that the functions $y_n$ be such as to make (10) imply

$$m^2(\lambda) \le \liminf_{S\to\infty} S^{-1} \int_0^S (\lambda - \mu + f)^2 \, dt, \qquad (\mu \ge 0). \tag{11}$$

(It is clear that the inequality $\mu \ge 0$ in (11), and not merely $\mu > 0$, can be allowed.) Now suppose that $\lambda \ge -c_0$ and choose $\mu \ge 0$ so that $\lambda - \mu = -c_0$. Then (11), (1), and the Parseval relation yield

$$m^2(\lambda) \le \int_0^1 (-c_0 + f)^2 \, dt = 2 \sum_{n=1}^\infty | c_n |^2,$$

so that the relation (5) is now proved.

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# AN ADDITION TO PORITSKY'S SOLUTIONS OF A DIFFERENTIAL EQUATION OF TORSION*

By J. C. WILHOIT, JR. (*Stanford University*)

In the theory of circular shafts of variable cross section, the only stress components which do not vanish are the shearing stresses,

$$P_{x\theta} = \frac{G}{r^2}\frac{\partial\psi}{\partial r}, \qquad P_{r\theta} = -\frac{G}{r^2}\frac{\partial\psi}{\partial x} \tag{1}$$

the stress function $\psi$ is a function of $r$ and $x$ alone, and the coordinate system is cylindrical $(r, x, \theta)$. Equations (1) satisfy the equilibrium equations and will satisfy compatibility if

$$\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial x}\left(\frac{1}{r^3}\frac{\partial\psi}{\partial x}\right) = 0 \tag{2a}$$

or

$$\frac{\partial^2\psi}{\partial r^2} - \frac{3}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial x^2} = 0. \tag{2b}$$

In problems relating to cones or spheres, it will be convenient to express Eq. (2a) in spherical coordinates $R$, $\omega$, $\phi$, as

$$\frac{\partial}{\partial R}\left[\frac{1}{R^2}\frac{1}{(1-\mu^2)^2}\frac{\partial\psi}{\partial R}\right] + \frac{\partial}{\partial\mu}\left[\frac{1}{R^4}\frac{1}{(1-\mu^2)}\frac{\partial\psi}{\partial\mu}\right] = 0, \tag{3}$$

where $R$ is the radius in spherical coordinates and $\mu = \cos\omega = x/R$.

The product solutions of (3) are:

$$\psi = (1-\mu^2)^2 P''_{n+1}(\mu) R^{-n}, \tag{4a}$$

$$\psi = (1-\mu^2)^2 P''_{n+1}(\mu) R^{n+3}, \tag{4b}$$

$$\psi = (1-\mu^2)^2 Q''_{n+1}(\mu) R^{-n}, \tag{4c}$$

$$\psi = (1-\mu^2)^2 Q''_{n+1}(\mu) R^{n+3}, \tag{4d}$$

where $Pn$ and $Qn$ are the Legendre functions of the first and second kind.

The solutions (4) do not give functions $\psi$ varying as $R$, and only Eq. (4d) gives a solution varying as $R^2 (n = -1)$. These missing solutions are:

$$\psi = (1+\mu^2)R^2 \qquad \psi = R^2\mu, \tag{5a}$$

$$\psi = (1+\mu^2)R \qquad \psi = R\mu. \tag{5b}$$

Solutions (4) are given by equations (79) and (80) of reference [1]. The solutions (5a) and (5b), however, do not agree with those given in Eqs. (107) and (108) of reference [1].

Although one is usually interested in torsion of shafts with stress free surfaces, it is

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interesting to consider the type of loading which results from Eqs. (1) if this requirement is relaxed. For a cone, consideration of the equilibrium of a small volume element at the surface of the cone will show that the surface stress consists of rings of shear with a magnitude

$$P_{\omega\phi} = \frac{1}{r^2}\frac{d\psi}{dR}\bigg|_{\mu=\text{const.}} = \frac{1}{R^2(1-u^2)}\frac{d\psi}{dR}\bigg|_{\mu=\text{const.}}. \tag{6}$$

Equation (6) indicates that for a given shear distribution on the surface of the cone, the boundary condition is on $(d\psi/dR)_{\mu=\text{const.}}$ rather than on $\psi$. It is evident that although Eqs. (4) and (5) give a complete set of functions varying as any positive or negative power of $R$, this set does not include any function which has a derivative behaving as $1/R$. The two functions which have a derivative behaving as $1/R$ are, omitting arbitrary multipliers.

$$\psi_1 = \left[\frac{\mu^2}{2} + \frac{(3\mu - \mu^3)}{4}\log\frac{1+\mu}{1-\mu} + \frac{\log(1-\mu^2)}{2} + \log R\right],$$

$$\psi_2 = \left[\frac{5\mu}{3} - \frac{\mu^3}{9} - \frac{1}{3}\log\frac{1+\mu}{1-\mu} - \frac{(3\mu-\mu^3)}{6}\log(1-\mu^2) + \frac{(3\mu-\mu^3)}{3}\log R\right]. \tag{7}$$

To obtain these functions the following procedure was used:

1. Equation (3) is multiplied by $R^4$ and differentiated with respect to $R$ giving

$$\frac{\partial}{\partial R}\left[R^4\frac{\partial}{\partial R}\left(\frac{1}{R^2}\frac{1}{(1-\mu^2)^2}\psi_R\right) + \frac{\partial}{\partial\mu}\left[\frac{1}{(1-\mu^2)}\frac{\partial\psi_R}{\partial\mu}\right] = 0. \tag{8}$$

2. The desired solutions are those with $\psi_R$ behaving as $1/R$. Thus $\psi_R$ must be of the form.

$$\psi_R = \frac{f(\mu)}{R}. \tag{9}$$

Equation (8) becomes an ordinary differential equation in $f(u)$, and shows that

$$\psi_R = \left[A\left(\mu - \frac{\mu^3}{3}\right) + B\right]\frac{1}{R} \tag{10}$$

and therefore

$$\psi = \left[A\left(\mu - \frac{\mu^3}{3}\right) + B\right]\log R + g(\mu), \tag{11}$$

where $A$ and $B$ are constants of integration

3. With

$$\psi_1 = B\log R + g_1(\mu), \qquad \psi_2 = A\left(\mu - \frac{\mu^3}{3}\right)\log R + g_2(\mu), \tag{12}$$

substitution into Eq. (3) will give ordinary differential equations for the unknown functions $g_1(u)$ and $g_2(u)$. These may be integrated, giving

$$g_1(\mu) = \frac{3}{4}B\mu^2 + \frac{3}{4}B\int(1-\mu^2)\log\frac{1+\mu}{1-\mu}\,d\mu,$$

$$g_2(\mu) = A\mu - \frac{A}{2} \int (1 - \mu^2) \log (1 - \mu^2)\, d\mu. \tag{13}$$

If the integration indicated in Eq. (13) is performed and the values of $g_1(\mu)$ and $g_2(\mu)$ used in Eq. (12), the expressions given in Eq. (7) are obtained.

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# BOOK REVIEWS

*Description of a magnetic drum calculator*. By The Staff of the Computation Laboratory. Harvard University Press, Cambridge, 1952. 318 pp. $8.00

This book is one of a series put out by the staff of the Computation Laboratory of Harvard University and describes the Mark III calculator. This machine was completed in March of 1950 and was then moved to the Naval Proving Ground at Dahlgren, Virginia.

The book itself is a detailed description of this machine and is of principal interest to those persons who are immediately associated with the machine. It combines both an engineering and a mathematical description of the device.

The text is extremely well illustrated both with photographs and with schematics of the principal organs of the machine.

To illustrate the coding of problems for the machine there is a chapter which contains among other things the programming for four illustrative examples.

The text is undoubtedly an invaluable aid to those immediately concerned with the operation and programming of problems for the Mark III calculator.

H. H. Goldstine

*Introduction to the theory of plasticity for engineers*. By Oscar Hoffman and George Sachs. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1953. xib + 276 pp. $6.50.

As one would expect on the basis of the authors' well deserved reputation their book contains an excellent exposition of the technological application of plasticity. Major emphasis is placed on the approximate solutions to problems of rolling, extruding, drawing, etc. in which the material is assumed to be ideally plastic and the true three dimensional character of the flow is not taken into account. Appreciable space is also devoted to the elementary classical problems of the thick-walled shell and tube and to the rotating cylinder and disk. Although the text opens with a discussion of stress and strain tensors and considers a stress space, the discussion of stress-strain relations and experimental data is brief and is essentially confined to the maximum shear stress and octahedral shear stress criteria. The extensive modern literature on stress-strain relations in the plastic range is, in the main, ignored. Also, except for a short section on two-dimensional plastic flow problems, little of the classical mathematical theory of plasticity is treated. No mention is made of plastic waves nor of the theorems or applications of limit analysis and design. The latter would be especially useful in evaluating some of the results obtained in the approximate solutions which are treated.

D. C. Drucker

*Tensor calculus.* By Barry Spain. Oliver and Boyd, Edinburgh and London, Interscience Publishers, Inc., New York, 1953. viii + 125 pp. $1.55.

In this readable little book, the author manages to cover a surprisingly large amount of material—tensor algebra and differentiation, and introductions to differential geometry, elasticity, and relativity. This is accomplished by using a very concise style of writing in which some aspects of a subject are presented only formally. The book should be of value to both mathematicians and engineers.

The following material is covered in the text. Tensors are defined by their transformation laws, and addition, multiplication, contraction, and the quotient law are discussed. The metric tensor and the principal directions of a second order symmetric tensor in $n$-dimensional Riemannian space are studied. After formally introducing the Christoffel symbols, the covariant derivative is defined and its properties are determined. Geodesics, parallelism, and the curvature tensor of $n$-dimensional Riemannian space are discussed. In addition, a brief survey of three-dimensional differential geometry is given (Frenet formulas for curves in space, the normal vector of a surface, the second fundamental tensor of surface, etc.). In the section on elasticity, orthogonal transformations, rotations, infinitesimal strain, the compatibility relations, the stress tensor, Hooke's law, and the equilibrium relations are studied. A brief introduction is given to curvilinear coordinates and isotropic tensors. Finally, an introduction is given to the special theory of relativity and then the general theory, including the Schwartzschild line-element and the Einstein and De Sitter universes, is discussed.

N. Coburn

*The theory of homogeneous turbulence.* By G. K. Batchelor. The University Press, Cambridge, 1953. xi + 197 pp. $5.00.

This book gives an excellent account of the modern developments in the statistical theory of homogeneous turbulence. The author has purposely chosen to omit all work requiring a Lagrangian description, since the methods used are very much different from those in the Eulerian description. Problems like diffusion are therefore omitted in the present book. It is also clear from the title of the book that a treatment of shear flow is not to be found in the present volume. The omission of these two important aspects of turbulence might discourage some people chiefly interested in applications. However, it is generally agreed among workers in turbulence that a treatment of the statistical theory of homogeneous turbulence would lead to the best understanding of our established knowledge and basic concepts.

Chapter I gives a general description of the problems involved in the statistical theory of turbulence, and a brief history of the subject, beginning with Taylor's work of 1935. Chapter II describes the methods used in the mathematical representation of the field of turbulence, including a discussion of the method of taking averages and the relation between correlation and spectral tensors. Chapter III applies these concepts to the treatment of specific velocity correlations.

Discussion of the dynamical aspects of turbulence begins with Chapter IV, where a set of linear problems are collected and treated in some detail. In Chapter V, the general aspects of the decay of homogeneous turbulence are treated, including a discussion of the final period of deday, where linearization is again justified.

The universal equilibrium theory of A. N. Kolmogoroff is described in Chapter VI. General discussions of the energy transfer is included here. The specific spectrum proposed by Townsend is, however, postponed to Chapter VII where Heisenberg's form of the energy spectrum (as calculated by Chandrasekhar) is also discussed. The concept of self-preservation during the process of decay is treated in Chapter VII. The author presents some limited experimental information before introducing the general concept, although the general ideas were conceived and used theoretically for the prediction of law of decay before these detailed experimental information were available. It is quite likely that the general concepts will hold in other cases not yet examined experimentally.

The final chapter gives a discussion of the probability distribution of the velocity fluctuation and the consequences drawn from the approximately Gaussian character of the joint velocity distribution at several points. The book ends with an excellent bibliography.

C. C. Lin

*Advances in applied mechanics.* Volume III. By Richard von Mises and Theodore von Kármán. Academic Press Inc., New York, 1953. x + 324 pp. $9.00.

This volume contains the following articles.

*G. F. Carrier: Boundary layer problems in applied mechanics.* The term "boundary layer problem," borrowed from hydrodynamics, is here applied to problems leading to non-dimensional differential equations in which the coefficient of the highest derivative is small compared to the coefficients of lower derivatives. The solution of such a problem may often be obtained by matching a "boundary layer solution" to an "interior solution". Several examples are given that illustrate this technique.

*O. Zaldastani: The one-dimensional isentropic fluid flow.* The discussion is restricted to analytical solutions of the linear equations obtained by applying a Legendre transformation to the equation of continuity and the momentum equation for one-dimensional isentropic flow. Particular attention is given to the possible occurrence of singularities. The procedure of unfolding the characteristic plane that the author ascribes to Ludford in this connection is actually due to S. Christianovich (Mat. Sbornik *1*, 511, 1936). Interaction of simple waves, expansion of a monatomic gas into a vacuum, and flow in a closed tube are discussed as examples.

*F. N. Frenkiel: Turbulent mean diffusion: mean concentration distribution in a flow field of homogeneous turbulence.* On account of the importance of turbulent diffusion to chemical engineers, meteorologists, and other readers not expected to be familiar with the general theory of turbulence, the article begins with a concise discussion of the statistical description of a turbulent field. The main portion of the article is concerned with diffusion in a field of homogeneous isotropic turbulence in which the decay of turbulence is neglected. Diffusion from a point source and diffusion from a line source are discussed. An extension of the results to a case of non-isotropic turbulence is discussed in the appendix.

*H. F. Ludloff: On aerodynamics of blast.* The article treats the diffraction of weak shocks around wedges of arbitrary angle, the pressure and density fields behind blasts advancing over arbitrary flat surfaces, and the head-on encounter of a shock with a wall that is almost parallel to the shock front. Most space is devoted to the last two problems and the discussion is based on previously published work by the author, C. S. Gardner, and L. Ting.

*G. Guderley: On the presence of shocks in mixed subsonic-supersonic flow patterns.* A supersonic region that is embedded in a subsonic flow field will, in general, contain shocks. The first part of the article discusses this phenomenon from the standpoint of the mathematician and the second, from that of the physicist. The mathematical problem is attacked by studying the change that a small deformation of the boundaries produces in a transonic potential flow. It is shown that the resulting boundary value problem does not, in general, admit a continuous solution. The particular solutions that indicate the impossibility of a potential flow are then discussed physically. They represent wavy flow patterns that may lead to physically unacceptable overlapping portions of the flow field.

*L. Rosenhead: Vortex systems in wakes.* This, by far the shortest article in the volume, reviews the literature on vortex systems in wakes for the range of Reynolds numbers up to about 2500 and deals mainly with papers of a theoretical nature. The author concludes that "most of what has been done is interesting, suggestive, and qualitatively correct, but if greater precision is needed, it is necessary to reinvestigate in fine detail much of what has already been done."

*H. Geiringer: Some recent results in the theory of an ideal plastic body.* Most of this article is devoted to a thorough mathematical discussion of plane plastic flow. Arbitrary isotropic yield functions and plastic potentials are considered, and it is not assumed that the yield function and the plastic potential are identical. Much of the material may strike the superficial reader as well-known, but a closer inspection reveals a wealth of results that render more precise, amplify, or generalize, classical theorems. Unfortunately, the space available for this review does not allow detailed comments on this great number of small but valuable contributions to the mathematical theory of plasticity.

*A. I. Bellin: Non-autonomous systems.* The term "non-autonomous" is used for a system of one degree of freedom whose acceleration $a$ depends explicitly on the time $t$ and not only on the displacement $x$ and the velocity $v$. Particular attention is given to the case where $a(x,v,t)$ is a periodic function of $t$, and the behavior of the system is discussed with the aid of a three-dimensional phase space with the coordinates $x$, $v$, and $t$. Criteria for the stability of periodic solutions are presented. Important special cases are discussed in detail.

W. Prager

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